2 Sequences and Accumulation Points

2.1 Convergent Sequences

Formally, a SEQUENCE OF REAL NUMBERS is a function $\varphi : \mathbb{N} \to \mathbb{R}$. For instance the function $\varphi(n) = \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$ defines a sequence[3]. It is customary, though, to write sequences by listing their terms such as

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$$

or by writing them in the form $(a_n)_{n\in\mathbb{N}}$; so in our particular example we could write the sequence also as $\left(\dfrac{1}{n^2}\right)_{n\in\mathbb{N}}$. Note that the elements of a sequence come in a natural order. For instance $\dfrac{1}{49}$ is the 7th element of the sequence $\left(\dfrac{1}{n^2}\right)_{n\in\mathbb{N}}$.

Exercise 2.1
Let $(a_n)_{n\in\mathbb{N}}$ denote the sequence of prime numbers in their natural order. What is a_5?[4]

Exercise 2.2
Write the sequence $0, 1, 0, 2, 0, 3, 0, 4, \ldots$ as a function $\varphi : \mathbb{N} \to \mathbb{R}$.

We say that a sequence (a_n) IS CONVERGENT, if there is a real number a, such that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$,

$$|a_n - a| < \varepsilon.$$

The number a is called LIMIT of the sequence (a_n). We also say in this case that the sequence $(a_n)_{n\in\mathbb{N}}$ CONVERGES TO a.

[3] See Figure 19 on page 75 for a pronunciation guide of Greek letters.
[4] "a_5" is pronounced "a sub 5".

Exercise 2.3
Spend some quality time studying Figure 1 on the next page. Explain how the pictures and the parts in the definition correspond to each other. Also reflect on how the "rigorous" definition above relates to your prior understanding of what it means for a sequence to converge.

A sequence, which fails to converge, is called DIVERGENT. Figure 2 on page 12 gives an example.

Exercise 2.4

1. Write down formally (using ε-N language) what it means that a given sequence $(a_n)_{n\in\mathbb{N}}$ does not converge to the real number a.
2. Similarly, write down what it means for a sequence to diverge.

Exercise 2.5
Show that the sequence $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ converges to 0.

Exercise 2.6
Show that the sequence $a_n = 1 - \dfrac{1}{n^2+1}$ converges to 1.

The first general result below establishes that limits are unique.

Task 2.7
Show: If a sequence converges to two real numbers a and b, then $a = b$.

One way to proceed is to assume that the sequence converges to two numbers a and b with $a \neq b$. Then one tries to derive a contradiction, since far out sequence terms must be simultaneously close to both a and b.

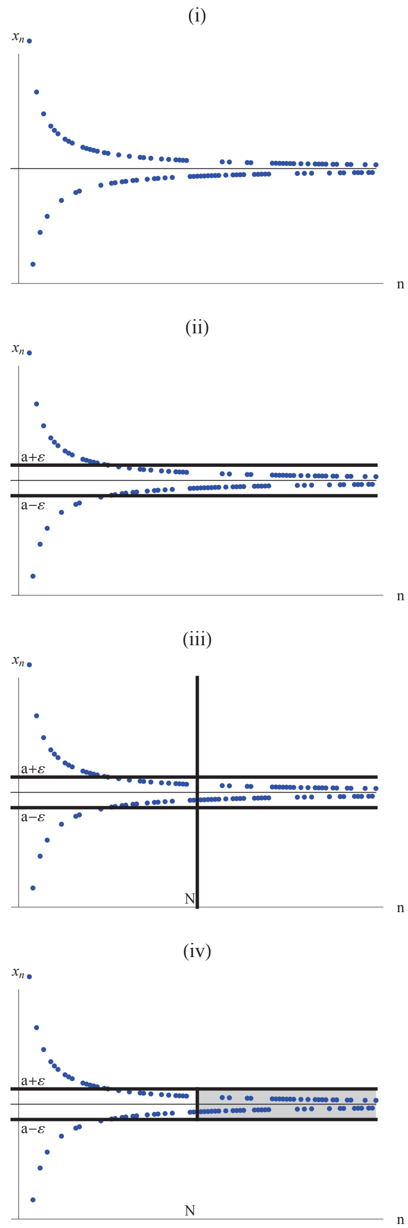

Figure 1: (i) A sequence (x_n) converges to the limit a if ... (ii) ... for all $\varepsilon > 0$ ... (iii) ... there is an $N \in \mathbb{N}$, such that ... (iv) ... $|x_n - a| < \varepsilon$ for all $n \geq N$

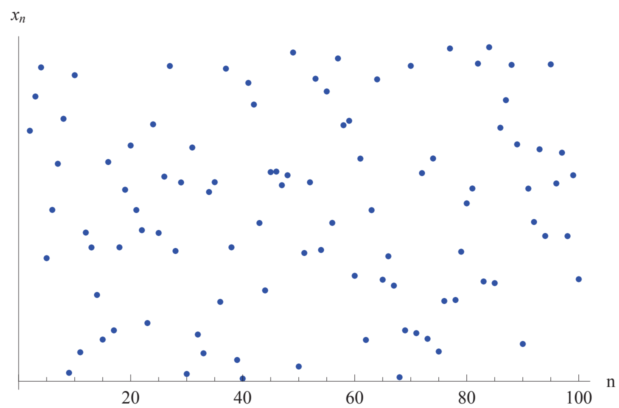

Figure 2: A divergent sequence

We say that a set S of real numbers is BOUNDED if there are real numbers m and M such that

$$m \leq s \leq M$$

holds for all $s \in S$.

A sequence (a_n) is called BOUNDED if its range

$$\{a_n \mid n \in \mathbb{N}\}$$

is a bounded set.

Exercise 2.8
Give an example of a bounded sequence which does not converge.

Task 2.9
Every convergent sequence is bounded.

Consequently, boundedness is necessary for convergence of a sequence, but is not sufficient to ensure that a sequence is convergent.

2.2 Arithmetic of Converging Sequences

The following results deal with the "arithmetic" of convergent sequences.

Task 2.10
If the sequence (a_n) converges to a, and the sequence (b_n) converges to b, then the sequence $(a_n + b_n)$ is also convergent and its limit is $a + b$.

Task 2.11
If the sequence (a_n) converges to a, and the sequence (b_n) converges to b, then the sequence $(a_n \cdot b_n)$ is also convergent and its limit is $a \cdot b$.

Task 2.12
Let (a_n) be a sequence converging to $a \neq 0$. Then there are a $\delta > 0$ and an $M \in \mathbb{N}$ such that $|a_m| > \delta$ for all $m \geq M$.

Task 2.12 is useful to prove:

Task 2.13
Let the sequence (b_n) with $b_n \neq 0$ for all $n \in \mathbb{N}$ converge to $b \neq 0$. Then the sequence $\left(\frac{1}{b_n}\right)$ is also convergent and its limit is $\frac{1}{b}$.

Task 2.14
Let (a_n) be a sequence converging to a. If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

2.3 Monotone Sequences

Let A be a non-empty set of real numbers. We say that A is BOUNDED FROM ABOVE if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in A$. The number M is then called an UPPER BOUND for A. Similarly, we say that A is BOUNDED FROM BELOW if there is an $m \in \mathbb{R}$ such that $a \geq m$ for all $a \in A$. The number m is then called a LOWER BOUND for A.

A sequence is BOUNDED FROM ABOVE (BOUNDED FROM BELOW), if its range

$$\{a_n \mid n \in \mathbb{N}\}$$

is bounded from above (bounded from below).

A sequence (a_n) is called INCREASING if $a_m \leq a_n$ for all $m < n \in \mathbb{N}$. It is called STRICTLY INCREASING if $a_m < a_n$ for all $m < n \in \mathbb{N}$.

Analogously, a sequence (a_n) is called DECREASING if $a_m \geq a_n$ for all $m < n \in \mathbb{N}$. It is called STRICTLY DECREASING if $a_m > a_n$ for all $m < n \in \mathbb{N}$.

A sequence which is increasing or decreasing is called MONOTONE.

The following axiom is a fundamental property of the real numbers. It establishes that bounded monotone sequences are convergent. Most results in Analysis depend on this fundamental axiom.

> **Completeness Axiom of the Real Numbers.** Let (a_n) be an increasing bounded sequence. Then (a_n) converges.

The same result holds of course if one replaces "increasing" by "decreasing". (Can you prove this?)

Note that an increasing sequence is always bounded from below, while a decreasing sequence is always bounded from above.

Task 2.15

Let $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 1}$ for all $n \in \mathbb{N}$. Show that the sequence (a_n) converges.

Once we know that the sequence converges, we can find its limit as follows: Let

$$L = \lim_{n \to \infty} a_n.$$

Then $\lim_{n\to\infty} a_{n+1} = L$ as well, and therefore $L = \lim_{n\to\infty} \sqrt{2a_n + 1} = \sqrt{2L+1}$. Since L is positive and L satisfies the equation

$$L = \sqrt{2L+1},$$

we conclude that the limit of the sequence under consideration is equal to $1 + \sqrt{2}$.

Let A be a non-empty set of real numbers. We say that a real number s is the LEAST UPPER BOUND of A (or that s is the SUPREMUM of A), if

1. s is an upper bound of A, and
2. no number smaller than s is an upper bound for A.

We write $s = \sup A$.

Similarly, we say that a real number i is the GREATEST LOWER BOUND of A (or that i is the INFIMUM of A), if

1. i is a lower bound of A, and
2. no number greater than i is a lower bound for A.

We write $i = \inf A$.

Exercise 2.16
Show the following: If a non-empty set A of real numbers has a maximum, then the maximum of A is also the supremum of A.

An INTERVAL I is a set of real numbers with the following property:

If $x \leq y$ and $x, y \in I$, then $z \in I$ for all $x \leq z \leq y$.

In particular, the set

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

is called a CLOSED INTERVAL. Similarly, the set

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

is called an OPEN INTERVAL.

Exercise 2.17
Find the supremum of each of the following sets:

1. The closed interval $[-2, 3]$
2. The open interval (0,2)
3. The set $\{x \in \mathbb{Z} \mid x^2 < 5\}$
4. The set $\{x \in \mathbb{Q} \mid x^2 < 3\}$.

Exercise 2.18
Let (a_n) be an increasing bounded sequence. By the Completeness Axiom the sequence converges to some real number a. Show that its range $\{a_n \mid n \in \mathbb{N}\}$ has a supremum, and that the supremum equals a.

The previous task uses the Completeness Axiom for the Real Numbers. Note that without using the Completeness Axiom we can still obtain the following weaker result: If an increasing bounded sequence converges, then it converges to the supremum of its range.

Task 2.19
The Completeness Axiom is equivalent to the following: Every non-empty set of real numbers which is bounded from above has a supremum.

The following hints may be useful to prove the "hard" direction of this result. The three hints are independent of each other and suggest different ways in which to proceed. Assume the non-empty set S is bounded from above.

- Suppose $a \in S$ and b is an upper bound of S. Then **(a)** there is an upper bound b' of S such that $|b' - a| \leq |b - a|/2$, or **(b)** there exists $a' \in S$ such that $|b - a'| \leq |b - a|/2$.
- Show that for all $\varepsilon > 0$ there is an element $a \in S$ such that $a + \varepsilon$ is an upper bound for S.

- Show that the set of upper bounds of S is of the form $[s, \infty)$. Then show that s is the supremum of S.

Optional Task 2.1
Use the Completeness Axiom to show the following: For every positive real number $s \in \mathbb{R}$, $s > 0$, there is a natural number $n \in \mathbb{N}$ such that $n - 1 \leq s < n$.

2.4 Subsequences

Recall that a sequence is a function $\varphi : \mathbb{N} \to \mathbb{R}$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function[5].

Then the sequence $\varphi \circ \psi : \mathbb{N} \to \mathbb{R}$ is called a SUBSEQUENCE of φ.

Here is an example: Suppose we are given the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots$$

The map $\psi(n) = 2n$ then defines the subsequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$$

If we denote the original sequence by (a_n), and if $\psi(k) = n_k$ for all $k \in \mathbb{N}$, then we denote the subsequence by (a_{n_k}).

So, in the example above,

$$a_{n_1} = a_2 = \frac{1}{2}, a_{n_2} = a_4 = \frac{1}{4}, a_{n_3} = a_6 = \frac{1}{6}, \ldots$$

Exercise 2.20
Let $(a_n)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}}$. Which of the following sequences are subsequences of $(a_n)_{n \in \mathbb{N}}$?

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$

[5] A function $\psi : \mathbb{N} \to \mathbb{N}$ is called strictly increasing if it satisfies: $\psi(n) < \psi(m)$ for all $n < m$ in $\mathbb{N}$.

2. $\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5} \ldots$

3. $1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15} \ldots$

4. $1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5} \ldots$

For the subsequence examples, also find the function $\psi : \mathbb{N} \to \mathbb{N}$.

Task 2.21

If a sequence converges, then all of its subsequences converge to the same limit.

Task 2.22

Show that every sequence of real numbers has an increasing subsequence or it has a decreasing subsequence.[6]

Does your proof of Task 2.22 actually show a slightly stronger result?

The next fundamental result is known as the **Bolzano-Weierstrass Theorem.**

Task 2.23

Every bounded sequence of real numbers has a convergent subsequence.

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[6]This result has a beautiful generalization due to Frank P. Ramsey (1903–1930), which unfortunately requires some notation: Given an infinite subset M of $\mathbb{N}$, we denote the set of doubletons from M by

$$\mathcal{P}^{(2)}(M) := \{\{m, n\} \mid m, n \in M \text{ and } m < n\}.$$

- **Ramsey's Theorem [24].** Let $\mathcal{A}$ be an arbitrary subset of $\mathcal{P}^{(2)}(\mathbb{N})$. Then there is an infinite subset M of $\mathbb{N}$ such that either

$$\mathcal{P}^{(2)}(M) \subseteq \mathcal{A}, \quad \text{or} \quad \mathcal{P}^{(2)}(M) \cap \mathcal{A} = \emptyset.$$

You should prove Task 2.22 without using this theorem, but the result in Task 2.22 follows easily from Ramsey's Theorem: Set

$$\mathcal{A} = \{\{m, n\} \mid m < n \text{ and } a_m \leq a_n\}.$$

Exercise 2.24
Let $a < b$. Every sequence contained in the interval $[a, b]$ has a subsequence that converges to an element in $[a, b]$.

Task 2.25
Suppose the sequence (a_n) does **not** converge to the real number L. Then there is an $\varepsilon > 0$ and a subsequence (a_{n_k}) of (a_n) such that

$$|a_{n_k} - L| \geq \varepsilon \text{ for all } k \in \mathbb{N}.$$

We conclude this section with a rather strange result: it establishes convergence of a bounded sequence without ever showing any convergence at all.

Task 2.26
Let (a_n) be a **bounded** sequence. Suppose all of its **convergent** subsequences converge to the same limit a. Then (a_n) itself converges to a.

2.5 Limes Inferior and Limes Superior*

Let (a_n) be a bounded sequence of real numbers. We define the LIMES INFERIOR[7] and LIMES SUPERIOR of the sequence as

$$\liminf_{n\to\infty} a_n := \lim_{k\to\infty} \left(\inf\{a_n \mid n \geq k\}\right),$$

and

$$\limsup_{n\to\infty} a_n := \lim_{k\to\infty} \left(\sup\{a_n \mid n \geq k\}\right).$$

[7] "limes" means limit in Latin.

Optional Task 2.2
Explain why the numbers $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ are well-defined[8] for every bounded sequence (a_n).

One can define the notions of lim sup and lim inf without knowing what a limit is:

Optional Task 2.3
Show that the limes inferior and the limes superior can also be defined as follows:

$$\liminf_{n\to\infty} a_n := \sup\left\{\inf\{a_n \mid n \geq k\} \mid k \in \mathbb{N}\right\},$$

and

$$\limsup_{n\to\infty} a_n := \inf\left\{\sup\{a_n \mid n \geq k\} \mid k \in \mathbb{N}\right\}.$$

Optional Task 2.4
Show that a bounded sequence (a_n) converges if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Optional Task 2.5
Let (a_n) be a bounded sequence of real numbers. Show that (a_n) has a subsequence that converges to $\limsup_{n\to\infty} a_n$.

Optional Task 2.6
Let (a_n) be a bounded sequence of real numbers, and let (a_{n_k}) be one of its con-

[8] An object is well-defined if it exists and is uniquely determined.

verging subsequences. Show that

$$\liminf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$

2.6 Cauchy Sequences

A sequence $(a_n)_{n\in\mathbb{N}}$ is called a CAUCHY SEQUENCE[9], if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m \geq N$ and $n \geq N$,

$$|a_m - a_n| < \varepsilon.$$

Informally speaking: a sequence is convergent, if far out all terms of the sequence are close to the limit; a sequence is a Cauchy sequence, if far out all terms of the sequence are close to each other.

We will establish in this section that a sequence converges if and only if it is a Cauchy sequence.

You may wonder why we bother to explore the concept of a Cauchy sequence when it turns out that Cauchy sequences are nothing else but convergent sequences. Answer: You can't show directly that a sequence converges without knowing its limit a priori. The concept of a Cauchy sequence on the other hand allows you to show convergence without knowing the limit of the sequence in question! This will nearly always be the situation when you study series of real numbers. The "Cauchy criterion" for series turns out to be one of most widely used tools to establish convergence of series.

Exercise 2.27
Every convergent sequence is a Cauchy sequence.

Exercise 2.28
Every Cauchy sequence is bounded.

[9]Named in honor of Augustin Louis Cauchy (1789–1857)

Task 2.29
If a Cauchy sequence has a converging subsequence with limit a, then the Cauchy sequence itself converges to a.

Task 2.30
Every Cauchy sequence is convergent.

Optional Task 2.7
Show that the following three versions of the Completeness Axiom are equivalent:

1. Every increasing bounded sequence of real numbers converges.
2. Every non-empty set of real numbers which is bounded from above has a supremum.
3. Every Cauchy sequence of real numbers converges.

From an abstract point of view, our course in "Analysis on the Real Line" hinges on two concepts:

- We can measure the **distance** between real numbers. More precisely we can measure "small" distances: for instance, we have constructs such as "$|a_n - a| < \varepsilon$" measuring how "close" a_n and a are.
- We can **order** real numbers. The statement "For all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$", for example, relies exclusively on our ability to order real numbers.

Thus the last two versions of the Completeness Axiom point to possible generalizations of our subject matter.

The second version uses the concept of **order** (boundedness, least upper bound), but does not mention distance. Section 2.5 gives clues how to define the limit concept in such a scenario.

The last version of the Completeness Axiom, on the other hand, requires the ability to measure small **distances**, but does not rely on order. This will be useful when defining completeness for sets such as $\mathbb{C}$ and $\mathbb{R}^n$ that cannot be ordered in the same way real numbers can.

2.7 Accumulation Points

Given $x \in \mathbb{R}$ and $\varepsilon > 0$, we say that the open interval $(x-\varepsilon, x+\varepsilon)$ forms a NEIGHBORHOOD OF x.

We say that a property $P(n)$ holds FOR ALL BUT FINITELY MANY $n \in \mathbb{N}$ if the set $\{n \in \mathbb{N} \mid P(n) \text{ does not hold}\}$ is finite.

Task 2.31
A sequence (a_n) converges to $L \in \mathbb{R}$ if and only if every neighborhood of L contains all but a finite number of the terms of the sequence (a_n).

The real number x is called an ACCUMULATION POINT of the set S, if every neighborhood of x contains infinitely many elements of S.

Task 2.32
The real number x is an accumulation point of the set S if and only if every neighborhood of x contains an element of S different from x.

Note that finite sets do not have accumulation points.

The following exercise provides some more examples:

Exercise 2.33
Find all accumulation points of the following sets:

1. $\mathbb{Q}$
2. $\mathbb{N}$
3. $[a, b)$

4. $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.

Exercise 2.34

1. Find a set of real numbers with exactly two accumulation points.
2. Find a set of real numbers whose accumulation points form a sequence (a_n) with $a_n \neq a_m$ for all $n \neq m$.

Task 2.35

Show that x is an accumulation point of the set S if and only if there is a sequence (x_n) of elements in S with $x_n \neq x_m$ for all $n \neq m$ such that (x_n) converges to x.

Task 2.36

Every infinite bounded set of real numbers has at least one accumulation point.

Optional Task 2.8

Characterize all infinite sets that have no accumulation points.

The next tasks in this section explore the relationship between the limit of a converging sequence and accumulation points of its range.

Optional Task 2.9

1. Find a converging sequence whose range has exactly one accumulation point.
2. Find a converging sequence whose range has no accumulation points.

3. Show that the range of a converging sequence has at most one accumulation point.

Optional Task 2.10
Suppose the sequence (a_n) is bounded and satisfies the condition that $a_m \neq a_n$ for all $m \neq n \in \mathbb{N}$. If its range $\{a_n \mid n \in \mathbb{N}\}$ has exactly one accumulation point a, then (a_n) converges to a.

The remaining tasks investigate how accumulation points behave with respect to some of the usual operations of set theory.

For any set S of real numbers, we denote by $\mathbf{A}(S)$ the set of all accumulation points of S.

Optional Task 2.11
If S and T are two sets of real numbers and if $S \subseteq T$, then $\mathbf{A}(S) \subseteq \mathbf{A}(T)$.

Optional Task 2.12
If S and T are two sets of real numbers, then

$$\mathbf{A}(S \cup T) = \mathbf{A}(S) \cup \mathbf{A}(T).$$

Optional Task 2.13
1. Let $(S_n)_{n\in\mathbb{N}}$ be a collection of sets of real numbers. Show that
$$\bigcup_{n\in\mathbb{N}} \mathbf{A}(S_n) \subseteq \mathbf{A}\left(\bigcup_{n\in\mathbb{N}} S_n\right).$$
2. Find a collection $(S_n)_{n\in\mathbb{N}}$ of sets of real numbers such that
$$\bigcup_{n\in\mathbb{N}} \mathbf{A}(S_n) \quad \text{is a proper subset of} \quad \mathbf{A}\left(\bigcup_{n\in\mathbb{N}} S_n\right).$$

Optional Task 2.14

Let S be a set of real numbers. Show that $\mathbf{A}(\mathbf{A}(S)) \subseteq \mathbf{A}(S)$.