4 Continuity

4.1 Definition and Examples

Let D be a set of real numbers and $x_0 \in D$. A function $f : D \to \mathbb{R}$ is said to be CONTINUOUS at x_0 if the following holds: For all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in D$ with

$$|x - x_0| < \delta,$$

we have that

$$|f(x) - f(x_0)| < \varepsilon.$$

If the function is continuous at all $x_0 \in D$, we simply say that $f : D \to \mathbb{R}$ is continuous on D.

Compare this definition of continuity to the earlier definition of having a limit. For continuity, we want to ensure that the behavior of the function close to the point x_0 nicely interacts with the behavior of the function at the point in question itself; thus we require that x_0 lies in the domain D, and that the "limit" equals $f(x_0)$. Note also that we do no longer require in the definition of continuity that x_0 is an accumulation point of D.

Exercise 4.1
Let D be a set of real numbers and $x_0 \in D$ be an accumulation point of D. Then the function $f : D \to \mathbb{R}$ is continuous at x_0 if and only if $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Exercise 4.2
Let D be a set of real numbers and $x_0 \in D$. Assume also that x_0 is not an accumulation point of D. Then the function $f : D \to \mathbb{R}$ is continuous at x_0.

Optional Task 4.1
Let D be a set of real numbers and $x_0 \in D$. A function $f : D \to \mathbb{R}$ is continuous at x_0 if and only if for all sequences (x_n) in D converging to x_0, the sequence $(f(x_n))$ converges to $f(x_0)$.

Exercise 4.3

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} |x|, & \text{if } x \in \mathbb{Q} \\ x^2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 is $f(x)$ continuous?

Exercise 4.4

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ continuous at $x_0 = 0$? See Figure 3 on page 27.

Exercise 4.5

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ continuous at $x_0 = 0$?

See Figure 5 on page 30.

Exercise 4.6

Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 is $f(x)$ continuous?

Exercise 4.7
Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \text{ with } p, q \text{ relatively prime positive integers} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

For which values of x_0 is $f(x)$ continuous? See Figure 6 on page 31.

It is interesting to note that in the late 1890s René-Louis Baire (1874–1932) proved a beautiful result which implies that there are no functions on the real line that are continuous at all rational numbers and discontinuous at all irrational numbers.

4.2 Combinations of Continuous Functions

Optional Task 4.2
Let $D \subseteq \mathbb{R}$, and let $f, g : D \to \mathbb{R}$ be functions continuous at $x_0 \in D$. Then $f + g : D \to \mathbb{R}$ is continuous at x_0.

Optional Task 4.3
Let $D \subseteq \mathbb{R}$, and let $f, g : D \to \mathbb{R}$ be functions continuous at $x_0 \in D$. Then $f \cdot g : D \to \mathbb{R}$ is continuous at x_0.

Optional Task 4.4
Polynomials are continuous on $\mathbb{R}$.

Optional Task 4.5
Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function continuous at $x_0 \in D$. Assume additionally that $f(x) \neq 0$ for all $x \in D$. Then $\dfrac{1}{f} : D \to \mathbb{R}$ is continuous at x_0.

Task 4.8
Let $D, E \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function continuous at $x_0 \in D$. Assume $f(D) \subseteq E$. Suppose $g : E \to \mathbb{R}$ is a function continuous at $f(x_0)$. Then the composition $g \circ f : D \to \mathbb{R}$ is continuous at x_0.

4.3 Uniform Continuity

We say that a function $f : D \to \mathbb{R}$ is UNIFORMLY CONTINUOUS on D if the following holds: For all $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x, y \in D$ satisfy

$$|x - y| < \delta,$$

then

$$|f(x) - f(y)| < \varepsilon.$$

Exercise 4.9
If $f : D \to \mathbb{R}$ is uniformly continuous on D, then f is continuous on D. What is the difference between continuity and uniform continuity?

Exercise 4.10
Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{x}$. Show that f is not uniformly continuous on $(0, 1)$.

Similarly one can show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$, is continuous on $\mathbb{R}$, but fails to be uniformly continuous on $\mathbb{R}$.

Task 4.11
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Show that f is uniformly continuous on $[a, b]$.

Along the way, you probably want to use the Bolzano-Weierstrass Theorem (Task 2.24 on page 19) to prove this result.

In light of Exercise 4.10, the result of Task 4.11 must depend heavily on properties of the domain. It is therefore natural to ask for what domains continuity automatically implies uniform continuity. The following two tasks explore this question.

Optional Task 4.6
Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrary function. Then f is uniformly continuous on $\mathbb{N}$.

Optional Task 4.7
Let D be a set of real numbers. Give a characterization of all the domains D such that every continuous function $f : D \to \mathbb{R}$ is uniformly continuous on D. [13]

Task 4.12
Let $f : D \to \mathbb{R}$ be uniformly continuous on D. If D is a bounded subset of $\mathbb{R}$, then $f(D)$ is also bounded.

Optional Task 4.8
Let $f : D \to \mathbb{R}$ be uniformly continuous on D. If (x_n) is a Cauchy sequence in D, then $(f(x_n))$ is also a Cauchy sequence.

Note a subtle, but important difference between the conclusion of the Task above and the characterization of continuity in Exercise 4.1: Even though every Cauchy sequence of elements in D will converge to some real number, that real number will not necessarily lie in D.

Optional Task 4.9
If a function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on the open interval (a, b), then it can be defined at the endpoints a and b in such a way that the extension $f : [a, b] \to \mathbb{R}$ is (uniformly) continuous on the closed interval $[a, b]$.

Thus, for instance, the function $f : (0,1) \to \mathbb{R}$, given by $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0,1)$.

It is often easier to show uniform continuity by establishing the following stronger condition:

A function $f : D \to \mathbb{R}$ is called a LIPSCHITZ FUNCTION on D if there is an $M > 0$ such that for all $x, y \in D$

$$|f(x) - f(y)| \leq M|x - y|$$

Exercise 4.13
Let $f : D \to \mathbb{R}$ be a Lipschitz function on D. Then f is uniformly continuous on D.

Task 4.14
Show: The function $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[0,1]$, but it is not a Lipschitz function on the interval $[0,1]$.

4.4 Continuous Functions on Closed Intervals

The major goal of this section is to show that the continuous image of a closed bounded interval is a closed bounded interval.

We say a function $f : D \to \mathbb{R}$ is BOUNDED, if there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$.

Exercise 4.15
Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$. Then f is bounded on $[a,b]$.

Optional Task 4.10
Let D be a set of real numbers. Give a characterization of all the domains D such that every continuous function $f : D \to \mathbb{R}$ is automatically bounded on D. [11]

We say that the function $f : D \to \mathbb{R}$ has an ABSOLUTE MAXIMUM if there exists an $x_0 \in D$ such that $f(x) \leq f(x_0)$ for all $x \in D$. Similarly, $f : D \to \mathbb{R}$ has an ABSOLUTE MINIMUM if there exists an $x_0 \in D$ such that $f(x) \geq f(x_0)$ for all $x \in D$.

We can improve upon the result of Exercise 4.15 as follows:

Task 4.16
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then f has an absolute maximum (and an absolute minimum) on $[a, b]$.

The next result is called the **Intermediate Value Theorem**.

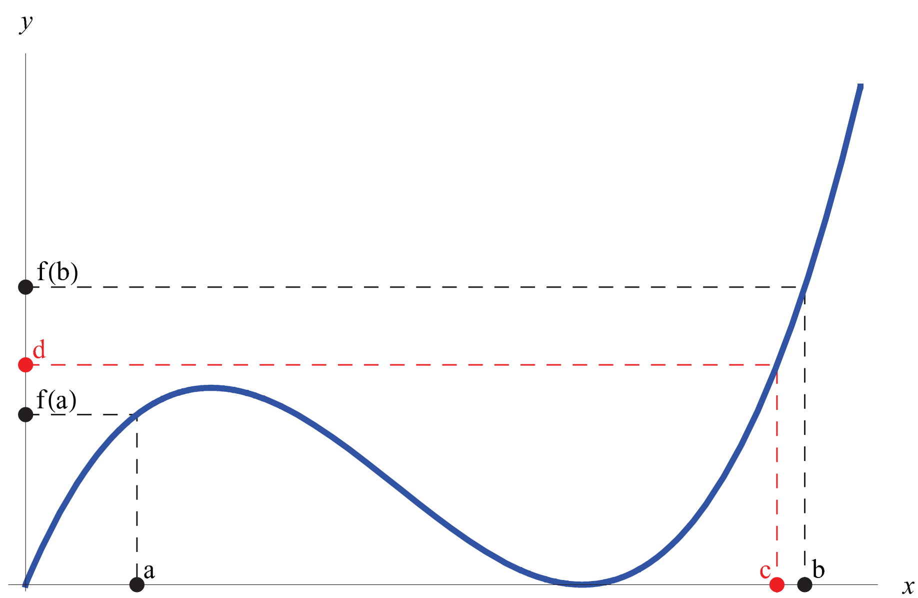

Figure 8: The Intermediate Value Theorem

Here the interval I can be any interval. Also: If $x > y$, we understand the interval (x, y) to be the interval (y, x).

Task 4.17
Let $f : I \to \mathbb{R}$ be a continuous function on the interval I. Let $a, b \in I$. If

$d \in (f(a), f(b))$, then there is a real number $c \in (a, b)$ such that $f(c) = d$. See Figure 8 on the preceding page.

A continuous function maps a closed bounded interval onto a closed bounded interval:

Task 4.18
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then $f([a, b]) := \{f(x) \mid x \in [a, b]\}$ is also a closed bounded interval.

Task 4.19
Let $f : [a, b] \to \mathbb{R}$ be strictly increasing (or decreasing, resp.) and continuous on $[a, b]$. Show that f has an inverse on $f([a, b])$, which is strictly increasing (or decreasing, resp.) and continuous.

Task 2.27 may be helpful to prove this result.

Task 4.20
Show that $\sqrt{x} : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$.