Math 3341 **Final Exam** **Spring 2020**

You must not use the class notes, your own notes or any other resource (book, internet, person, etc.) during the test. The final has 8 problems (for a total of 80 points) and 1 extra-credit problem on 9 pages. Good luck!

Problem 1 (10 points)

1. Let (a_n) be a sequence in $\mathbb{R}$, and let $a \in \mathbb{R}$. State the definition of "the sequence (a_n) converges to a".

2. Show that the sequence (a_n), where $a_n = \dfrac{2n+5}{3n-1}$ for all $n \in \mathbb{N}$, converges. What is the limit?

Problem 2 (10 points)

1. Define what it means for a sequence to be a Cauchy sequence.

2. Show from first principles that the product of two Cauchy sequences is a Cauchy sequence. You may use the fact that Cauchy sequences are bounded.

Problem 3 (10 points) Let (a_n) be a sequence converging to two real numbers a and b. Show that $a = b$ (uniqueness of limits).

Problem 4 (10 points) 1. Explain what it means that the real number x is an accumulation point of the set X.

2. Find all accumulation points of the set $\left\{\frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N}\right\}$. (No proof necessary.)

Problem 5 (10 points) 1. Give the precise definition for "limit of a function at a point."

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$. Show that f has a limit at $x_0 = 0$ (What is the limit?), or show that f does not have a limit at $x_0 = 0$.

Problem 6 (10 points)

1. State the Completeness Axiom.

2. Show that the Completeness Axiom implies that every decreasing bounded sequence converges.

Problem 7 (10 points) 1. Let $D \subseteq \mathbb{R}$, $x_0 \in D$, and $f : D \to \mathbb{R}$ be a function. Give the definition for "f is continuous at x_0".

2. Show: If f is continuous at x_0, then for every sequence (x_n) of elements in D that converges to x_0, the sequence $f(x_n)$ converges to $f(x_0)$.

Problem 8 (10 points)

1. Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. Define what it means that f is uniformly continuous on D.

2. Let $a < b$ be two real numbers. Show: If a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Extra Credit Problem 9 (10 points) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is "vanishing at ∞", if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|f(x)| \leq \varepsilon$ for all $|x| \geq N$.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and vanishes at ∞. Show that f is uniformly continuous on $\mathbb{R}$.