1. The salary scales in three school districts are as follows, for a teacher with a master’s degree:

District P: $30,000 plus $1,500 for each year of experience

District Q: $30,000 plus $1,750 for each year of experience

District R: $28,000 plus $1,750 for each year of experience

1. Give a formula for the salary in each district for a teacher with *n* years of experience

$$S\_{P}=1500n+30000; S\_{Q}=1750n+30000; S\_{R}=1750n+28000$$

1. Use your formulas to indicate the number of years experience teachers have in districts P and R when they earn the same salary.

$$1500n+30000=1750n+28000$$

$$⇔2000=250n$$

$$⇔n=8$$

1. Use your formulas to indicate the number of years experience teachers have in districts P and Q when they earn the same salary

$$1500n+30000=1750n+30000$$

$$⇔0=250n$$

$$⇔n=0$$

1. Use your formulas to indicate the number of years experience teachers have in districts Q and R when they earn the same salary

$$1750n+30000=1750n+28000$$

$$⇔30000=28000$$

$⇒$ Teachers from districts Q and R will never earn the same salary since $30000\ne 28000$

1. If in District T, teachers ear a salary $S\_{T}$ dollars plus $E\_{T}$ dollars for each year of experience, and in District U, teachers earn a salary of $S\_{U}$ dollars plus $E\_{U}$ dollars for each year of experience, $S\_{T}>S\_{U}$ and $E\_{U}>E\_{T}$, how many years will it take District U teachers to catch up to District T.

$$S\_{T}+E\_{T}n=S\_{U}+E\_{U}n$$

$$⇔S\_{T}-S\_{U}=E\_{u}n-E\_{T}n=n(E\_{U}-E\_{T})$$

$$⇔\frac{S\_{T}-S\_{U}}{E\_{U}-E\_{T}}=n$$

$⇒$ It would take district U $\frac{S\_{T}-S\_{U}}{E\_{U}-E\_{T}}$ years to catch up to district T provided $S\_{T}>S\_{U}$ and $E\_{U}>E\_{T}$

1. Let $A=\left[\begin{matrix}a&b\\c&d\end{matrix}\right],M=\left[\begin{matrix}e&f\\g&h\end{matrix}\right],$ and $X=\left[\begin{matrix}w&x\\y&z\end{matrix}\right].$ Show that $\left(A⋅M\right)⋅X=A⋅\left(M⋅X\right),$ and thus prove that multiplication of $2×2$ matrices is associative.

$$\left(A⋅M\right)⋅X=\left(\left[\begin{matrix}a&b\\c&d\end{matrix}\right]⋅\left[\begin{matrix}e&f\\g&h\end{matrix}\right]\right)⋅\left[\begin{matrix}w&x\\y&z\end{matrix}\right]=\left[\begin{matrix}ae+bg&af+bh\\ce+dg&cf+dh\end{matrix}\right]⋅\left[\begin{matrix}w&x\\y&z\end{matrix}\right]=\left[\begin{matrix}\left(ae+bg\right)w+\left(af+bh\right)y&\left(ae+bg\right)x+\left(af+bh\right)z\\\left(ce+dg\right)w+\left(cf+dh\right)y&\left(ce+dg\right)x+\left(cf+dh\right)z\end{matrix}\right] $$

$$A⋅\left(M⋅X\right)=\left[\begin{matrix}a&b\\c&d\end{matrix}\right]⋅\left(\left[\begin{matrix}e&f\\g&h\end{matrix}\right]⋅\left[\begin{matrix}w&x\\y&z\end{matrix}\right]\right)=\left[\begin{matrix}a&b\\c&d\end{matrix}\right]⋅\left[\begin{matrix}ew+fy&ex+fz\\gw+hy&gx+hz\end{matrix}\right]$$

$$=\left[\begin{matrix}a\left(ew+fy\right)+b\left(gw+hy\right)&a\left(ex+fz\right)+b\left(gx+hz\right)\\c\left(ew+fy\right)+d\left(gw+hy\right)&c\left(ex+fz\right)+d\left(gx+hz\right)\end{matrix}\right]$$

$$=\left[\begin{matrix}aew+afy+bgw+bhy&aex+afz+bgx+bhz\\cew+cfy+dgw+dhy&cex+cfz+dgx+dhz\end{matrix}\right]$$

$$=\left[\begin{matrix}\left(ae+bg\right)w+\left(af+bh\right)y&\left(ae+bg\right)x+\left(af+bh\right)z\\\left(ce+dg\right)w+\left(cf+dh\right)y&\left(ce+dg\right)x+\left(cf+dh\right)z\end{matrix}\right]=\left(A⋅M\right)⋅X$$

1. a. Show steps in solving the systems $\left\{\begin{array}{c}ωa+xc=1\\ya+zc=0\end{array}\right.$ and $\left\{\begin{array}{c}wb+xd=0\\yb+zd=1\end{array}\right.$ for *w*, *x*, *y*, and *z* in terms of *a*, *b*, *c*, and *d*, without using matrices.
2. Rewrite the systems of equations as $\left\{\begin{array}{c}x=\frac{1-wa}{c}\\y=\frac{-zc}{a}\end{array}\right.$ and $\left\{\begin{array}{c}x=\frac{-wb}{d}\\y=\frac{1-zd}{b}\end{array}\right.$
(You could also choose to write them in terms of *x* and *y* WLOG)
3. Set the first equations from both systems equal to each other and solve for *w*

$$\frac{1-wa}{c}=\frac{-wb}{d}$$

$$⇔d\left(1-wa\right)=-cwb$$

$$⇔d-dwa=-cwb$$

$$⇔cwb-dwa=-d$$

$$⇔w\left(cb-da\right)=-d$$

$$⇔w=\frac{-d}{cb-da}=\frac{d}{ad-bc}$$

1. Set the second equations from both systems equal to each other and solve for *z*

$$\frac{-zc}{a}=\frac{1-zd}{b}$$

$$⇔-bzc=a\left(1-zd\right)=a-azd$$

$$⇔azd-bzc=a$$

$$⇔z\left(ad-bc\right)=a$$

$$⇔z=\frac{a}{ad-bc}$$

1. Plug the value for *w* into one of the original first equations of a system and solve for *x*.
(It does not matter which one or that you use the original. However, the calculations are simpler this way.)

$$wa+xc=1⇔\left(\frac{d}{ad-bc}\right)a+xc=1$$

$$⇔\frac{ad}{ad-bc}-1=-xc$$

$$⇔\frac{ad}{ad-bc}-\frac{ad-bc}{ad-bc}=-xc$$

$$⇔\frac{ad-ad-bc}{ad-bc}=-xc$$

$$⇔xc=\frac{-bc}{ad-bc}$$

$$⇔x=\frac{-b}{ad-bc}$$

1. Plug the value for *z* into one of the original second equations of a system and solve for *y*.
(It does not matter which one or that you use the original. However the calculations are simpler this way.)

$$ya+zc=0⇔ya+\left(\frac{a}{ad-bc}\right)c=0$$

$$⇔ya=\frac{-ca}{ad-bc}$$

$$⇔y=\frac{-c}{ad-bc}$$

1. b. Explain why part **a** determines the multiplicative inverse of a $2×2$ matrix

The inverse of a $2×2$ matrix $\left[\begin{matrix}a&b\\c&d\end{matrix}\right]$exists if and only if there exists a matrix $\left[\begin{matrix}w&x\\y&z\end{matrix}\right]$ such that $\left[\begin{matrix}w&x\\y&z\end{matrix}\right]\left[\begin{matrix}a&b\\c&d\end{matrix}\right]=I=\left[\begin{matrix}1&0\\0&1\end{matrix}\right]$ by the definition of a multiplicative inverse matrix. When these two matrices are multiplied together, you get the systems of equations presented in part **a**. Since part **a** determines the values of *w*, *x*, *y*, and *z* are when $\left[\begin{matrix}w&x\\y&z\end{matrix}\right]\left[\begin{matrix}a&b\\c&d\end{matrix}\right]=I$ and those values make up the inverse of a general $2×2$ matrix, part **a** determines the multiplicative inverse of a $2×2$ matrix.

1. Use $2×2$ matrices to solve each system for $(x,y)$.
2. $\left\{\begin{array}{c}2x-5y=11\\4x-15y=-4\end{array}\right.$

$$\left[\begin{matrix}2&-5\\4&-15\end{matrix}\right]\left[\begin{matrix}x\\y\end{matrix}\right]=\left[\begin{matrix}11\\-4\end{matrix}\right]$$

$$⇒\left[\begin{matrix}x\\y\end{matrix}\right]=\left[\begin{matrix}2&-5\\4&-15\end{matrix}\right]^{-1}⋅\left[\begin{matrix}11\\-4\end{matrix}\right]$$

$$⇔\left[\begin{matrix}x\\y\end{matrix}\right]=\frac{1}{2\left(-15\right)-\left(-5\right)\left(4\right)}\left[\begin{matrix}-15&5\\-4&2\end{matrix}\right]⋅\left[\begin{matrix}11\\-4\end{matrix}\right]$$

$$=\frac{1}{-30+20}\left[\begin{matrix}\left(-15\right)\left(11\right)+5\left(-4\right)\\\left(-4\right)\left(11\right)+2\left(-4\right)\end{matrix}\right]$$

$$=\frac{1}{-10}\left[\begin{matrix}-165-20\\-44-8\end{matrix}\right]=-\frac{1}{10}\left[\begin{matrix}-18.5\\-5.2\end{matrix}\right] $$

$⇒x=\frac{-185}{-10}=18.5$ and $y=\frac{-52}{-10}=5.2$

1. $\left\{\begin{array}{c}ax+by=e\\cx+dy=f\end{array}\right.$, assuming $ad-bc\ne 0$

$$\left[\begin{matrix}a&b\\c&d\end{matrix}\right]\left[\begin{matrix}e\\f\end{matrix}\right]=\left[\begin{matrix}x\\y\end{matrix}\right]$$

$$⇒\left[\begin{matrix}x\\y\end{matrix}\right]=\left[\begin{matrix}a&b\\c&d\end{matrix}\right]^{-1}\left[\begin{matrix}e\\f\end{matrix}\right]$$

$$⇔\left[\begin{matrix}x\\y\end{matrix}\right]=\frac{1}{ad-bc}\left[\begin{matrix}d&-b\\-c&a\end{matrix}\right]\left[\begin{matrix}e\\f\end{matrix}\right]=\frac{1}{ad-bc}\left[\begin{matrix}de-bf\\-ce+af\end{matrix}\right]$$

$⇒x=\frac{de-bf}{ad-bc}$ and $y=\frac{-ce+af}{ad-bc}$, provided $ad-bc\ne 0$