Math 4303 Test 2 Spring 2020

This is an open test: you are free to use any resources you want. **Prohibited** is any contact with classmates or other people during the exam. **Show all your work!**
You need to email the test back to me by 16:25. PDF preferred, photos accepted if necessary.
The test has five problems on five pages.

Problem 1 (20 points) 1. Let M be the set of real-valued 2×2 matrices with determinant -1:

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad ad - bc = -1 \right\}.$$

Find matrices $A, B \in M$ such that the equation $A \cdot X = B$ does not have a solution $X \in M$.

Let A be any matrix in M. The equation $A \cdot X = A$ has a unique solution, namely the identity matrix. But $Id_2 \notin M$ since $\det Id_2 = +1$.

More generally $(M, \cdot)$ is not a group, since it does not have a neutral element, f.i.

2. Let N be the set of real-valued 2×2 matrices with determinant 1. Does every equation $A \cdot X = B$, with $A, B \in N$, have a unique solution $X \in N$? Explain!

$(N, \cdot)$ is a group! $Id_2 \in N$ is the neutral element, and since $\det(A^{-1}) = (\det A)^{-1}$, N is closed under inversion:
$A \in N \Rightarrow A^{-1} \in N$.

Thus every equation $A \cdot X = B$ has a unique solution in N.

Problem 2 (20 points)

Find the rule, the largest possible domain and the range for the function whose graph is the parabola passing through the points $(-1, -5)$, $(0, -4)$, and $(1, -1)$.

The general functional rule for a quadratic function is $P(x) = ax^2 + bx + c$

Using the information given we obtain 3 linear equations

(1) $a - b + c = -5$

(2) $\boxed{c = -4}$

(3) $a + b + c = -1$

(2) $-$ (1) yields $2b = 4 \Rightarrow \boxed{b = 2}$

Finally (3) yields. $\boxed{a = -1 - b - c = 1}$

so $P(x) = x^2 + 2x - 4 \quad (= (x+1)^2 - 5)$

the parabola is concave up, with its vertex at $x = -1 \Rightarrow f(-1) = -5$

Thus range $P = [-5, \infty)$

Of course, dom $P = \mathbb{R}$.

Problem 3 (20 points) 1. Find all real solutions in exact form of the equation

$$(x^2 + 4x + 4)^2 = -(x^2 + 5x + 6)^2$$

Explain carefully why your algebraic solving procedure is justified.

$\iff (x^2+4x4)^2+(x^2+5x+6)=0$

$\iff x^2+4x+4=0 \;\wedge\; x^2+5x+6=0$

$\iff (x+2)^2=0 \;\wedge\; (x+2)(x+3)=0$

$\iff (x+2=0) \wedge (x+2=0 \vee x+3=0)$

$\iff (x=-2) \wedge ((x=-2) \vee (x=-3))$

$\iff \underline{\underline{x=-2}}$

2. Find all real solutions in exact form of the equation

$$\ln(21 - x^2) = \ln(1 - x)$$

Explain carefully why your algebraic solving procedure is justified.

The domain of the equation is the interval $(-\sqrt{21}, 1)$
$\ln(x)$ is an 1-1 function on its domain so

$\ln(21-x^2) = \ln(1-x)$

$\iff 21-x^2 = 1-x$

$\iff x^2-x-20=0$

$\iff (x-5)(x+4)=0$

$\iff x=5$ or $x=-4$

Only $\underline{\underline{x=-4}}$ lies in the domain of the equation.

Problem 4 (20 points) 1. Find all real numbers x that solve the inequality

$$\frac{x^3 + 10x^2 + 25x}{x-5} \le 0.$$

Explain carefully why your algebraic solving procedure is justified.

$x^3 + 10x^2 + 25 = 0 \iff x(x+5)^2 = 0$

undefined

test points yield: $+ \quad + \quad - \quad +$

$-5 \qquad 0 \qquad 5$

Thus the solution set is: $\{-5\} \cup [0, 5)$

2. Find all real numbers x that solve the inequality

$$\log_x 4 > \log_{2x} 16.$$

Explain carefully why your algebraic solving procedure is justified.

I will assume that $x > 1$. Then $f(y) = (2x)^y$ and $g(y) = x^y$ are strictly increasing functions.

$\log_x 4 > \log_{2x} 16$

$\iff (2x)^{\log_x 4} > (2x)^{\log_{2x} 16}$

$\iff 2^{\log_x 4} \cdot 4 > 16 \iff 2^{\log_x 4} > 4 = 2^2$

since $h(x) = \cancel{\log_x 4}\ 2^x$ is strictly increasing

$\iff \log_x 4 > 2$

$\iff 4 > x^2$

$\iff -2 < x < 2$

Solution set: $x \in (1, 2)$

Problem 5 (20 points) The set I of isometries of the complex plane forms a group with the binary operation of composition. Consider the following two elements in I:

$$g(z) = 3i + \bar{z}i, \qquad h(z) = 5 - zi.$$

1. Check that h is indeed an isometry.

$$|h(z) - h(w)| = |(5 - zi) - (5 - wi)|$$
$$= |-zi + wi| = |-i||z - w| = |z - w|$$

2. Find the unique $f \in I$ such that $g \circ f = h$. Check that your f is an element in I.

We need to find $f \in I$ such that

$$g(f(z)) = h(z)$$
$$\Leftrightarrow \quad 3i + \overline{f(z)}\, i = 5 - zi$$
$$\Leftrightarrow \quad 3 + \overline{f(z)} = -5i \cancel{\mp} z$$
$$\Leftrightarrow \quad \overline{f(z)} = (-3 - 5i) \cancel{\mp} z$$
$$\Leftrightarrow \quad f(z) = (-3 + 5i) \cancel{\mp} \overline{z}$$
$$f(z) = (-3 + 5i) - \overline{z}$$

this is in the form of an isometry of Type II.