## Math 4303

Worksheet #1

January 23, 2020

Suppose that addition and multiplication have already been defined for the set of natural numbers  $\mathbb{N}$ .

1. We define a relation  $\sim$  on  $\mathbb{N} \times \mathbb{N}$  as follows:

$$(p,q) \sim (p',q') \Leftrightarrow p+q'=p'+q.$$

Show that  $\sim$  defines an equivalence relation on  $\mathbb{N} \times \mathbb{N}$ .

2. It then makes sense to define equivalence classes  $(p,q)_{\sim}$ :

$$(p,q)_{\sim} := \{ (p',q') \in \mathbb{N} \times \mathbb{N} \mid (p',q') \sim (p,q) \}.$$

The set of all these equivalence classes is denoted by  $(\mathbb{N} \times \mathbb{N})_{\sim}$ .

Find all elements in the equivalence class  $(2,5)_{\sim}$ . What do all these pairs of natural numbers have in common?

Find all elements in the equivalence class  $(4, 2)_{\sim}$ . What do all these pairs of natural numbers have in common?

- 3. One can then identify the set of integers Z with this set (N×N)<sub>∼</sub>. Which equivalence class corresponds to the integer 0? What about the equivalence classes corresponding to the integers 1 and -3, respectively?
- 4. How can one define addition of two integers? More precisely, what should be the meaning of

$$(p,q)_{\sim} + (p',q')_{\sim}?$$

Is your definition well-defined<sup>1</sup>?

- 5. Show that addition as defined in 4. is commutative.
- 6. What is the neutral element in  $(\mathbb{N} \times \mathbb{N})_{\sim}$  with respect to addition?
- 7. Given  $(p,q)_{\sim} \in (\mathbb{N} \times \mathbb{N})_{\sim}$ , what is the inverse element of  $(p,q)_{\sim}$  with respect to addition?
- 8. How can one define multiplication of two integers? More precisely, what should be the meaning of

$$(p,q)_{\sim} \cdot (p',q')_{\sim}?$$

Is your definition well-defined?

9. Verify that  $(2,5)_{\sim} \cdot (1,2)_{\sim} = (5,2)_{\sim}$ .

 $<sup>^{1}</sup>$ You have to check that your definition does not depend on the representatives chosen from the two equivalence classes.