Math 2326 **Laboratory 1A**

A Substitution Technique

You will explore a substitution technique which transforms certain differential equations into linear differential equations, which can then be solved by the method discussed in class.

1 Warm-up Exercise. Find the general solution of the following differential equations:

1. $y'(t) + y(t) = e^t$
2. $y'(t) + ty(t) = t$
3. $(1 + t^2)y'(t) - 2ty(t) = 2t(1 + t^2)$
4. $y'(t) + (\cot t)y(t) = 2 \csc t$

For $y > 0$ consider the differential equation

$$y'(t) + \frac{2}{t}y(t) = \frac{1}{t^2}(y(t))^3. \tag{1}$$

This differential equation is not linear because of the $y(t)^3$ term. We will now perform a substitution: we will replace $y(t)$ in the equation by

$$z(t) = y(t)^{-2},$$

or equivalently

$$y(t) = z(t)^{-1/2}.$$

We will also have to replace $y'(t)$ by an expression containing $z(t)$ and $z'(t)$, using the chain rule:

$$y'(t) = (-1/2)z(t)^{-3/2}z'(t).$$

Performing the substitution the differential equation (1) reads as:

$$(-1/2)z(t)^{-3/2}z'(t) + \frac{2}{t}z(t)^{-1/2} = \frac{1}{t^2}z(t)^{-3/2}. \tag{2}$$

Multiplying by the reciprocal of the term in front of $z'(t)$, we obtain the **linear** *differential equation:*

$$z'(t) - \frac{4}{t}z(t) = -\frac{2}{t^2}. \tag{3}$$

2 Show that equation (3) has the solutions $z(t) = \dfrac{2}{5t} + Ct^4$

Finally we re-substitute: The solutions to the original equation (1) *are given by*

$$y(t) = z(t)^{-1/2} = \left(\frac{2}{5t} + Ct^4\right)^{-1/2}.$$

3 Show that the substitution $z(t) = y(t)^{1-n}$ transforms a differential equation of the form

$$y'(t) + p(t)y(t) = q(t)y(t)^n \tag{4}$$

into a linear differential equation (in $z(t)$) for $n \neq 0, 1$!

4 Find the general solution of equation (4) for the cases $n = 0$ and $n = 1$.

A differential equation of the type $y'(t) + p(t)y(t) = q(t)y(t)^n$ is called a **Bernoulli Equation**.

5 Solve the following differential equations for the initial condition $y(1) = 2$:

1. $y'(t) - y(t) = -y(t)^4$
2. $t^2y'(t) - ty(t) = y(t)^2$
3. $y'(t) = \dfrac{1}{y(t)} - \dfrac{y(t)}{t+2}$

6 Solve the differential equations in **5** for the initial condition $y(1) = -2$. (You must slightly adapt the method for this initial condition! Why?)

7 The method of solving Bernoulli equations does not work when the initial condition is chosen so that $y = 0$. Why? (This does not mean necessarily that a solution does not exist.) What can you say about the solutions to the differential equations in **5** satisfying the initial condition $y(1) = 0$?