A Substitution Technique

You will explore a substitution technique which transforms certain differential equations into separable differential equations, which can then be solved by the method discussed in class.

1 Warm-up Exercise. Find the general solution of the following differential equations. Express the solutions in explicit form, whenever possible.

1. $\dfrac{dy}{dt} = \dfrac{3t^2 y}{1+t^3}$
2. $\dfrac{dy}{dt} = \dfrac{5y}{t(y+2)}$
3. $\dfrac{dy}{dt} = \sqrt{1-y^2}$
4. $\dfrac{dy}{dt} = \dfrac{1}{y\sqrt{1-y^2}}$

For $t > 0$ consider the differential equation

$$\frac{dy}{dt} = \frac{y^2 + ty}{t^2} \tag{1}$$

This differential equation is neither separable nor linear. We will now perform a substitution: we will replace $y(t)$ in the equation by

$$z(t) = \frac{y(t)}{t},$$

or equivalently

$$y(t) = tz(t).$$

We will also have to replace $\dfrac{dy}{dt}$ by an expression containing $z(t)$ and $z'(t)$, using the product rule:

$$\frac{dy}{dt} = z + t\frac{dz}{dt}.$$

Performing the substitution on both sides of the differential equation (1) *reads as:*

$$z + t\frac{dz}{dt} = z^2 + z. \tag{2}$$

Simplifying, we obtain the **separable** *differential equation:*

$$t\frac{dz}{dt} = z^2. \tag{3}$$

2 Show that equation (3) has the solutions $z(t) = \dfrac{1}{C - \ln t}$

Finally we re-substitute: The solutions to the original equation (1) *are given by*

$$y(t) = tz(t) = \frac{t}{C - \ln t}.$$

What is special about the differential equation (1)*? It can be written in the form* $\dfrac{dy}{dt} = F\left(\dfrac{y}{t}\right)$*, i.e. the right hand side depends only on the* **quotient of** y **and** t *and not on* y *and* t *independently.*

3 What is this function F for the differential equation (1)?

Differential equations of the form $\dfrac{dy}{dt} = F\left(\dfrac{y}{t}\right)$ *are called* **homogeneous**.

4 Show that the substitution $z(t) = \dfrac{y(t)}{t}$ transforms a homogeneous differential equation into a separable differential equation (in $z(t)$)!

5 Solve the following differential equations. Express the solutions in explicit form if possible.

1. $\dfrac{dy}{dt} = \dfrac{t^2 + ty + y^2}{t^2}$
2. $\dfrac{dy}{dt} = \dfrac{y - t}{y + t}$
3. $\dfrac{dy}{dt} = -\dfrac{t^2 + y^2}{ty}$

4. $\dfrac{dy}{dt} = \dfrac{y}{t} + \sin\left(\dfrac{y}{t}\right)$

6 Investigate the slope fields of homogeneous differential equations: How can you tell from the slope field that a given differential equation is homogeneous? **Hint**: Find the lines on which the slope is constant.

7 Investigate the slope field of the differential equation

$$\frac{dy}{dt} = \frac{y-t+3}{y+t-1}. \tag{4}$$

Using your observations in **6**, explain why the differential equation is not homogeneous. Compare the slope field to the slope field of the homogeneous equation

$$\frac{dy}{dt} = \frac{y-t}{y+t}. \tag{5}$$

Find a substitution method to transform the differential equation (4) into the homogeneous differential equation (5), then solve differential equation (4).