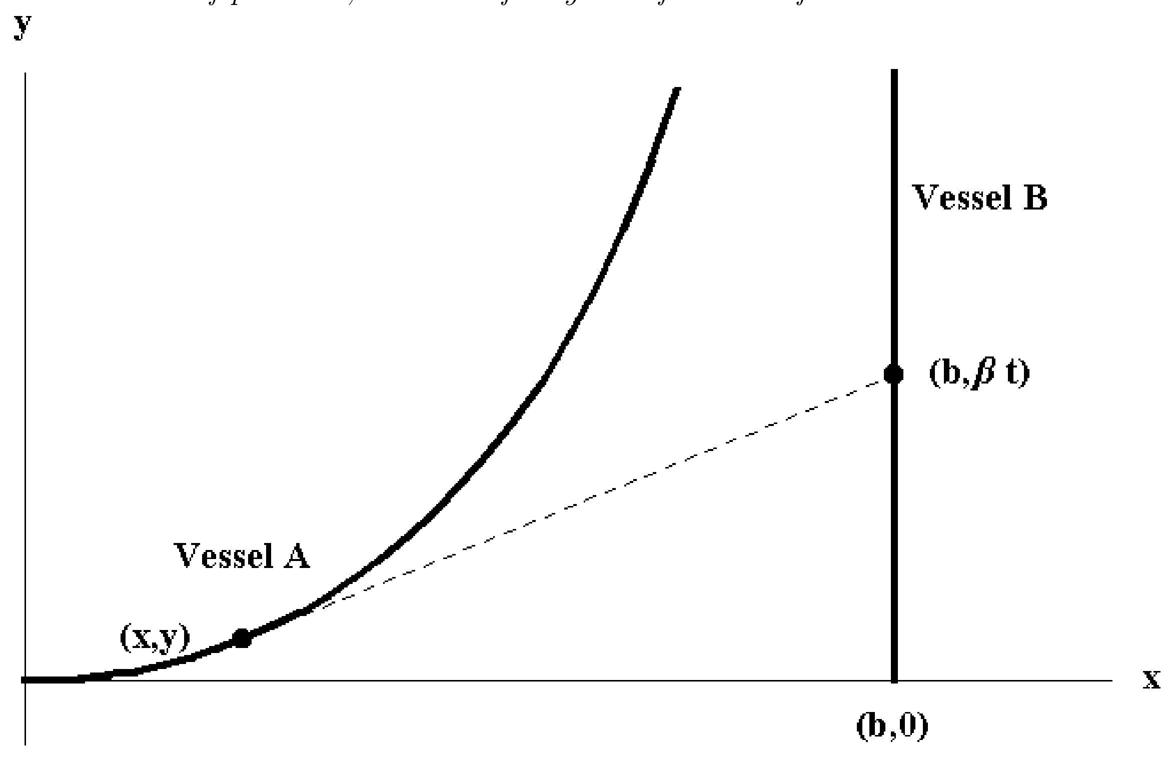
The Curve of Pursuit[1]

An interesting geometric problem—first considered by Leonardo da Vinci—arises when one tries to determine the path of a pursuer chasing its prey. The simplest problem is to find the curve along which a vessel moves in pursuing another vessel that flees along a straight line, assuming that the speeds of both vessels are constant.

Let's assume vessel A, traveling at a speed α, is pursuing vessel B, which is traveling at speed β. In addition, assume vessel A begins at time $t = 0$ at the origin and pursues vessel B, which begins at the point $(b, 0)$, $b > 0$, and travels up the line $x = b$. After t hours, vessel A is located at the point $P = (x, y)$ and vessel B is located at the point $Q = (b, \beta t)$. The goal is to describe the locus of points P; that is to find y as a function of x.



[1]This laboratory is based on a group project in *"Fundamentals of Differential Equations"* by R. Kent Nagle and Edward B. Saff.

1. Vessel A is pursuing vessel B, so at time t, vessel A must be heading right at vessel B. Thus the tangent line to the curve of pursuit at P must pass through the point Q. For this to be true, show that

$$\frac{dy}{dx} = \frac{y - \beta t}{x - b}.$$

2. We know that the speed at which vessel A is traveling, so we know that the distance it travels in time t is αt. This distance is also the length of the pursuit curve from $(0, 0)$ to (x, y). Using the arclength formula, show that

$$\alpha t = \int_0^x \sqrt{1 + \left[\frac{dy}{du}\right]^2}\, du.$$

3. Solve the formulas in **1.** and **2.** for the variable t to obtain

$$\frac{y}{\beta} - \frac{(x-b)}{\beta}\frac{dy}{dx} = \frac{1}{\alpha}\int_0^x \sqrt{1 + \left[\frac{dy}{du}\right]^2}\, du.$$

4. Set $w(x) = dy/dx$ and differentiate both sides of the equation above with respect to x to obtain the first-order differential equation

$$(x-b)\frac{dw}{dx} = -\frac{\beta}{\alpha}\sqrt{1+w^2}.$$

5. Using appropriate initial values for both x and $w = dy/dx$ when $t = 0$, show that the solution of the equation in **4.** is given by

$$\frac{dy}{dx} = w(x) = \frac{1}{2}\left[\left(1 - \frac{x}{b}\right)^{-\frac{\beta}{\alpha}} - \left(1 - \frac{x}{b}\right)^{\frac{\beta}{\alpha}}\right].$$

6. For the case $\alpha > \beta$, integrate the expression in **5.** with respect to x to obtain y as a function of x. What is the initial condition for y when $t = 0$?

7. For the case $\alpha > \beta$, find the location where vessel A will intercept vessel B.

8. Repeat **6.** for the case $\alpha = \beta$.

9. For the case $\alpha = \beta$, will vessel A ever reach vessel B?

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