Aquaculture[1]
(Prerequisite: Laboratory 1A or 1B)

Aquaculture is the art of cultivating the plants and fish indigenous to water. In the example considered here, it is assumed that a batch of catfish are raised in a pond. We are interested in determining the best time for harvesting the fish so that the cost per pound for raising the fish is minimized.

A differential equation describing the growth of fish may be expressed as

$$\frac{dW}{dt} = kW^{\alpha},$$

where $W(t)$ is the weight of the fish at time t, and k and α are empirically determined growth constants. The functional form of this relationship is similar to growth models for other species. Modeling the growth rate or metabolic rate by a term like W^{α} is a common assumption. Biologists often refer to the equation above as the **allometric equation**. *It can be supported by plausibility arguments such as growth rate depending on the surface area of the gut (which varies like $W^{2/3}$) or depending on the volume of the animal (which varies like W).*

1. Solve the allometric equation for $\alpha \neq 1$.

2. The solution in **1.** grows large without bound, but in practice there is some limiting maximum weight M for the fish. This limiting weight may be included in the allometric equation by inserting a dimensionless variable S that can range between 0 and 1 and involves an empirically determined parameter μ.

Namely we now assume that

$$\frac{dW}{dt} = kW^{\alpha}S,$$

where S has the form

$$S = 1 - \left(\frac{W}{M}\right)^{\mu}.$$

[1] This laboratory is based on a group project in *"Fundamentals of Differential Equations"* by R. Kent Nagle and Edward B. Saff.

When $\mu = 1 - \alpha$, this equation becomes a **Bernoulli equation**, and has a closed form solution. Solve the equation, when $k = 12$, $\alpha = 0.75$, $\mu = 0.25$, $M = 81$ ounces and $W(0) = 1$ ounce. The constants are given for t measured in months.

3. The differential equation describing the cost $C(t)$ of raising fish for t months has 2 parameters: K_1 specifies the cost per month (due to costs such as interest, depreciation and labor); K_2, the cost for food, multiplies with the growth rate (because the amount of food consumed by the fish is approximately proportional to the growth rate).

That is

$$\frac{dC}{dt} = K_1 + K_2 \frac{dW}{dt}.$$

Solve this equation when $K_1 = 0.4$, $K_2 = 0.1$, $C(0) = \$1.10$, and $\frac{dW}{dt}$ as determined in **2.**

4. Explain why it is optimal to harvest the fish at the time when the ratio $\frac{C(t)}{W(t)}$ is at minimum. Estimate the optimal harvesting time to the nearest month.