1.1 The Natural Numbers

Definition. Richard Dedekind started by giving the following definition of the set of **Natural Numbers**[3]:

> *The natural numbers are a set* $\mathbb{N}$ *containing a special element called* 0*, and a function* $S : \mathbb{N} \to \mathbb{N}$ *satisfying the following axioms:*
>
> **(D1)** S *is injective*[4].
>
> **(D2)** $S(\mathbb{N}) = \mathbb{N} \setminus \{0\}$.[5]
>
> **(D3)** *If a subset* M *of* $\mathbb{N}$ *contains 0 and satisfies* $S(M) \subseteq M$*, then* $M = \mathbb{N}$.

The function S is called the successor function.

The first two axioms describe the process of counting, the third axiom assures the **Principle of Induction**:

Task 1.1
Let $P(n)$ be a predicate with the set of natural numbers as its domain. If

1. $P(0)$ is true, and
2. $P(S(n))$ is true, whenever $P(n)$ is true,

then $P(n)$ is true for all natural numbers.

---

[3] A similar definition of the natural numbers was introduced by Giuseppe Peano in 1889:

> *The natural numbers are a set* $\mathbb{N}$ *containing a special element called* 0*, and a function* $S : \mathbb{N} \to \mathbb{N}$ *satisfying the following axioms:*
>
> **(P1)** $0 \in \mathbb{N}$.
>
> **(P2)** *If* $n \in \mathbb{N}$*, then* $S(n) \in \mathbb{N}$.
>
> **(P3)** *If* $n \in \mathbb{N}$*, then* $S(n) \neq 0$.
>
> **(P4)** *If a set* A *contains* 0*, and if* A *contains* $S(n)$*, whenever it contains* n*, then the set* A *contains* $\mathbb{N}$.
>
> **(P5)** $S(m) = S(n)$ *implies* $m = n$ *for all* $m, n \in \mathbb{N}$.

[4] A function $f : A \to B$ is called *injective* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

[5] For a function $f : A \to B$, $f(A) := \{b \in B \mid f(a) = b \text{ for some } a \in A\}$.

Arithmetic Properties. **Addition** of natural numbers is established recursively in the following way: For a fixed but arbitrary $m \in \mathbb{N}$ we define

$$\begin{aligned} m + 0 &:= m \\ m + S(n) &:= S(m+n) \qquad \text{for all } n \in \mathbb{N} \end{aligned}$$

By Axiom (D3), adding n to the fixed m is then defined for all natural numbers n. It is not clear at this point that the recursive formula defines addition in a unique way. This will be proved later in Task 1.21.

Task 1.2
If we set $S(0) := 1$, then $S(m) = m + 1$ for all natural numbers $m \in \mathbb{N}$.

Use induction for the following:

Task 1.3
Show that addition on $\mathbb{N}$ is associative.

Task 1.4
Show that addition on $\mathbb{N}$ is commutative.

This last task implies in particular that 0 is the (unique) neutral element with respect to addition: $n + 0 = 0 + n = n$ holds for all $n \in \mathbb{N}$.

Here is the cancellation law for addition:

Task 1.5
If $m + k = n + k$, then $m = n$.

Multiplication of natural numbers is also defined recursively as follows: For a fixed but arbitrary $m \in \mathbb{N}$ we define

$$\begin{aligned} m \cdot 0 &:= 0 \\ m \cdot (n+1) &:= m \cdot n + m \qquad \text{for all } n \in \mathbb{N} \end{aligned}$$

Task 1.22 will show that this recursive formula defines multiplication in a unique manner.

Task 1.6
Show that the following distributive law holds for natural numbers:

$$(m + n) \cdot k = m \cdot k + n \cdot k.$$

Task 1.7
Show that 1 is the neutral element with respect to multiplication: For all natural numbers m,

$$m \cdot 1 = 1 \cdot m = m.$$

Task 1.8
Show that multiplication on $\mathbb{N}$ is commutative.

Task 1.9
Show that multiplication on $\mathbb{N}$ is associative.

Task 1.10
Show that multiplication is zero-divisor free:

$$m \cdot n = 0 \text{ implies } m = 0 \text{ or } n = 0.$$

Finally we can impose a **total order**[6] on $\mathbb{N}$ as follows: We say that $m \leq n$, if there is a natural number k, such that $m + k = n$.

Show that "$\leq$" is indeed a total order:

Task 1.11
"$\leq$" is reflexive[7].

Task 1.12
"$\leq$" is anti-symmetric[8].

[6] A relation $\sim$ on A is called a *total order*, if $\sim$ is reflexive, anti-symmetric, transitive, and has the property that for all $a, b \in A$, $a \sim b$ or $b \sim a$ holds.

[7] A relation $\sim$ on A is *reflexive* if for all $a \in A$, $a \sim a$.

[8] A relation $\sim$ on A is *anti-symmetric* if for all $a, b \in A$ the following holds: $a \sim b$ and $b \sim a$ implies that a=b.

Task 1.13
"$\leq$" is transitive[9].

Task 1.14
For all $m, n \in \mathbb{N}$, $m \leq n$ or $n \leq m$.

Show the following two compatibility laws:

Task 1.15
If $m \leq n$, then $m + k \leq n + k$ for all $k \in \mathbb{N}$.

Task 1.16
If $m \leq n$, then $m \cdot k \leq n \cdot k$ for all $k \in \mathbb{N}$.

Last not least, here is the cancellation law for multiplication:

[9] A relation $\sim$ on A is *transitive* if for all $a, b, c \in A$ the following holds: $a \sim b$ and $b \sim c$ implies that $a \sim c$.

Task 1.17
If $m \cdot k = n \cdot k$, then $m = n$ or $k = 0$.

Infinite Sets and the Existence of the Set of Natural Numbers. Do natural numbers exist? Following Dedekind, we will say that a set M is **infinite**, if there is an injective map $f : M \to M$ that is not surjective[10].

Task 1.18
Show that the set of natural numbers as defined on p. 2 is infinite.

Thus, the existence of the set of natural numbers implies the existence of infinite sets. In fact, we will show that the converse also holds:

Theorem. If there is an infinite set, then there is a model for the natural numbers.

<u>Proof:</u> Let A be an infinite set. Then there is a function $S : A \to A$ that is injective, but not surjective. Thus we can find an $a_0 \in A$ with $a_0 \notin S(A)$. Let

$$\mathcal{K} = \{B \subseteq A \mid a_0 \in B \text{ and } S(B) \subseteq B\}$$

Note that $A \in \mathcal{K}$, so $\mathcal{K} \neq \emptyset$. We set

$$N = \bigcap_{B \in \mathcal{K}} B.$$

Observe that $N \in \mathcal{K}$. Indeed, $a_0 \in N$, since $a_0 \in B$ for all $B \in \mathcal{K}$. Also

$$S(N) = S\left(\bigcap_{B \in \mathcal{K}} B\right) \subseteq \bigcap_{B \in \mathcal{K}} S(B) \subseteq \bigcap_{B \in \mathcal{K}} B = N.$$

[10]A function $f : A \to B$ is called *surjective*, if $f(A) = B$.

By its definition the set N is thus the smallest element of $\mathcal{K}$.

Finally we show that N with the function $S : N \to N$ (as successor function) and a_0 (in the role of 0) satisfies Axioms (D1)–(D3).

As the restriction of the injective function $S : A \to A$ to N, the function $S : N \to N$ is also injective. Thus (D1) is satisfied.

For (D2) we have to show that $S(N) = N \setminus \{a_0\}$. Since $a_0 \notin S(N)$ and $S(N) \subseteq N$, we obtain that $S(N) \subseteq N \setminus \{a_0\}$. For the remaining subset relation suppose to the contrary that there is a second element missing from the range of N: there is an element $n_0 \in N$ satisfying $n_0 \notin S(N)$ and $n_0 \neq a_0$. Set $N_0 = N \setminus \{n_0\}$. Note that $a_0 \in N_0$ and that $S(N_0) \subseteq N_0$. Thus $N_0 \in \mathcal{K}$. We also know that $N_0 \subsetneqq N$, yielding a contradiction.

Now let $M \subseteq N$, with $a_0 \in M$, and satisfying $S(M) \subseteq M$. Then $M \in \mathcal{K}$, and thus, again using the minimality of N in $\mathcal{K}$, it follows that $M \supseteq N$. This proves (D3) and completes the proof.

Task 1.19
Present the proof of this Theorem.

Recursion and Uniqueness. Before we give a proof of the "essential" uniqueness of the natural numbers, we will follow Dedekind and establish the following general **Recursion Principle**:

Task 1.20
Let A be an arbitrary set, and let $a \in A$ and a function $f : A \to A$ be given. Then there exists a unique map $\varphi : \mathbb{N} \to A$ satisfying

1. $\varphi(0) = a$, and
2. $\varphi \circ S = f \circ \varphi$.

Here is a possible outline for a proof: Consider all subsets $K \subseteq \mathbb{N} \times A$ with the following properties:

1. $(0, a) \in K$, and
2. If $(n, b) \in K$, then $(S(n), f(b)) \in K$.

Clearly $\mathbb{N} \times A$ itself has these properties; we can therefore define the smallest such set: Let

$$L = \bigcap \{K \subseteq \mathbb{N} \times A \mid K \text{ satisfies (1) and (2)}\}.$$

Now show by induction that for every $n \in \mathbb{N}$ there is a unique $b \in A$ with $(n, b) \in L$. This property defines φ by setting $\varphi(n) = b$ for all $n \in \mathbb{N}$.

The Recursion Principle makes it possible to define a recursive procedure (the function φ) via a formula (the function f).

Task 1.21
Define addition of an arbitrary natural number n and the fixed natural number m using the Recursion Principle.

Task 1.22
Define multiplication of an arbitrary natural number n with the fixed natural number m using the Recursion Principle.

Use the Recursion Principle to show that the set of natural numbers is unique in the following sense:

Task 1.23
Suppose that $\mathbb{N}$, $S : \mathbb{N} \to \mathbb{N}$ and 0 satisfy Axioms (D1)–(D3), and that $\mathbb{N}'$, $S' : \mathbb{N}' \to \mathbb{N}'$ and $0'$ satisfy Axioms (D1)–(D3) as well.

Then there is a bijection[11] $\varphi : \mathbb{N} \to \mathbb{N}'$ such that

1. $\varphi(0) = 0'$, and
2. $\varphi \circ S = S' \circ \varphi$.

[11] A function $f : A \to B$ is a *bijection*, if it is both injective and surjective.