Solutions

Math 2326 Test 3 November 22, 2022

You may use a non-graphing calculator. No cell phones or other internet-capable devices are permitted. Show your work to receive credit!
This test has 5 problems and an extra credit problem on 6 printed pages. Good luck!

Problem 1 (20 points) Solve the following initial value problem:

$$y'' - 4y' + 7y = 0 \qquad y(0) = 2, \quad y'(0) = 1.$$

char. eqn: $\lambda^2 - 4\lambda + 7 = 0$
with solutions
$\lambda = 2 \pm \sqrt{3}\, i$

general solution:
$y = Ae^{2t}\cos(\sqrt{3}t) + Be^{2t}\sin(\sqrt{3}t)$

$y(0) = 2 \Rightarrow A = 2$

$y' = 2Ae^{2t}\cos(\sqrt{3}t) - \sqrt{3}Ae^{2t}\sin(\sqrt{3}t)$
$+ 2Be^{2t}\sin(\sqrt{3}t) + \sqrt{3}Be^{2t}\cos(\sqrt{3}t)$

so $y'(0) = 2A + \sqrt{3}B$

$1 = y'(0) = 4 + \sqrt{3}B$

$B = \frac{-3}{\sqrt{3}} = -\sqrt{3}$

IVP solution:
$y(t) = 2e^{2t}\cos(\sqrt{3}t) - \sqrt{3}e^{2t}\sin(\sqrt{3}t)$

Problem 2 (20 points) For $b \geq 0$, and $k, m > 0$, the differential equation

$$y'' + \frac{b}{m}y' + \frac{k}{m}y = 0$$

describes the motion of a spring.

1. Suppose $b = 4$, $m = 1$, and $k = 2$. Find the general solution of the differential equation in this case.

$y'' + 4y' + 2y = 0$

charact. equ:

$\lambda^2 + 4\lambda + 2 = 0$

$\Rightarrow$ the solutions $\lambda = -2 \pm \sqrt{2}$

general solution:

$y(t) = Ae^{(-2+\sqrt{2})t} + Be^{-(2+\sqrt{2})t}$

2. Now suppose $b = 4$ and $m = 1$. For which values of k is the spring underdamped?

$y'' + 4y' + ky = 0$

$\Rightarrow$ the ch. equ.. $\lambda^2 + 4\lambda + k = 0$

$\lambda = -2 \pm \sqrt{4-k}$

Spring is underdamped if $4-k < 0$

$\Leftrightarrow k > 4$

Problem 3 (20 points) Consider the following system of non-linear differential equations:

$$\begin{aligned} x' &= 8 - x^2 - y^2 \qquad (1) \quad f(x,y) = 8 - x^2 - y^2 \\ y' &= x - y \qquad (2) \quad g(x,y) = x - y \end{aligned}$$

1. Find the two equilibrium points of this system.

(2) $x - y = 0 \Rightarrow y = x$

$\Rightarrow$ (1) $8 - 2x^2 = 0 \Leftrightarrow x^2 = 4$

Solutions: $(x,y) = (2,2)$ and $(x,y) = (-2,-2)$

2. Linearize the system at the equilibrium points, and classify the equilibrium points as *(spiral) sink*, *(spiral) source*, or *saddle*.

$$Jac = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x & -2y \\ 1 & -1 \end{pmatrix}$$

at $(2,2)$ $\quad Jac = \begin{pmatrix} -4 & -4 \\ 1 & -1 \end{pmatrix}$

ch. equ: $\lambda^2 + 5\lambda + 8 = 0$

so $\lambda = -\frac{5}{2} \pm \sqrt{\underbrace{\frac{25}{4} - 8}_{<0}}$ $\quad$ spiral sink

at $(-2,-2)$ $\quad Jac = \begin{pmatrix} 4 & 4 \\ 1 & -1 \end{pmatrix}$

with ch. equ. $\lambda^2 - 3\lambda - 8 = 0$

$\lambda = \frac{3}{2} \pm \sqrt{\underbrace{\frac{9}{4} + 8}_{> \frac{3}{2}}}$ $\quad$ saddle

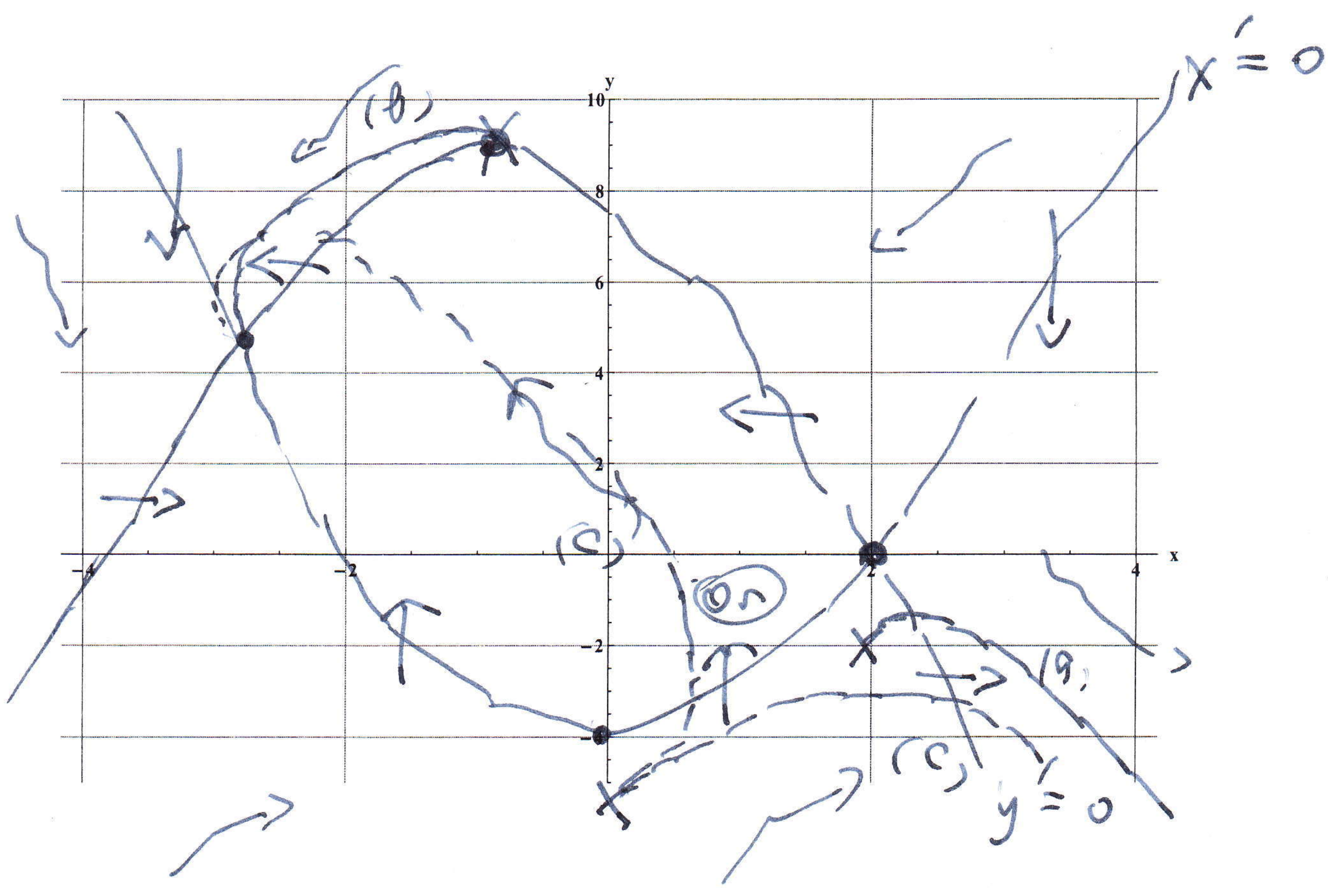
Problem 4 (~~15~~ 20 points) Consider the following system of non-linear differential equations:

$$\begin{aligned} x' &= x^2 - y - 4 \\ y' &= 9 - (x+1)^2 - y \end{aligned}$$

(The system has two equilibrium points: a saddle at the equilibrium point $(2,0)$ and a sink at the equilibrium point $(-3,5)$.)

1. Sketch the nullclines in the graph below.

2. Sketch the solutions with initial conditions **(a)** $(x_0, y_0) = (2, -1)$, **(b)** $(x_0, y_0) = (-1, 9)$, and **(c)** $(x_0, y_0) = (0, -5)$ for as long as they stay within the graph below.



Problem 5 (20 points) Consider the two-parameter family of systems of linear differential equations of the form

$$\mathbf{Y}'(t) = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix} \cdot \mathbf{Y}(t).$$

In the coordinate system below, clearly mark the regions in which the system has a sink, source, saddle, spiral sink, and spiral source, respectively.

ch. eqn: $\lambda^2 - 2a\lambda + (a^2 - b) = 0$

so $\lambda = a \pm \sqrt{a^2 - (a^2 - b)}$

$= a \pm \sqrt{b}$

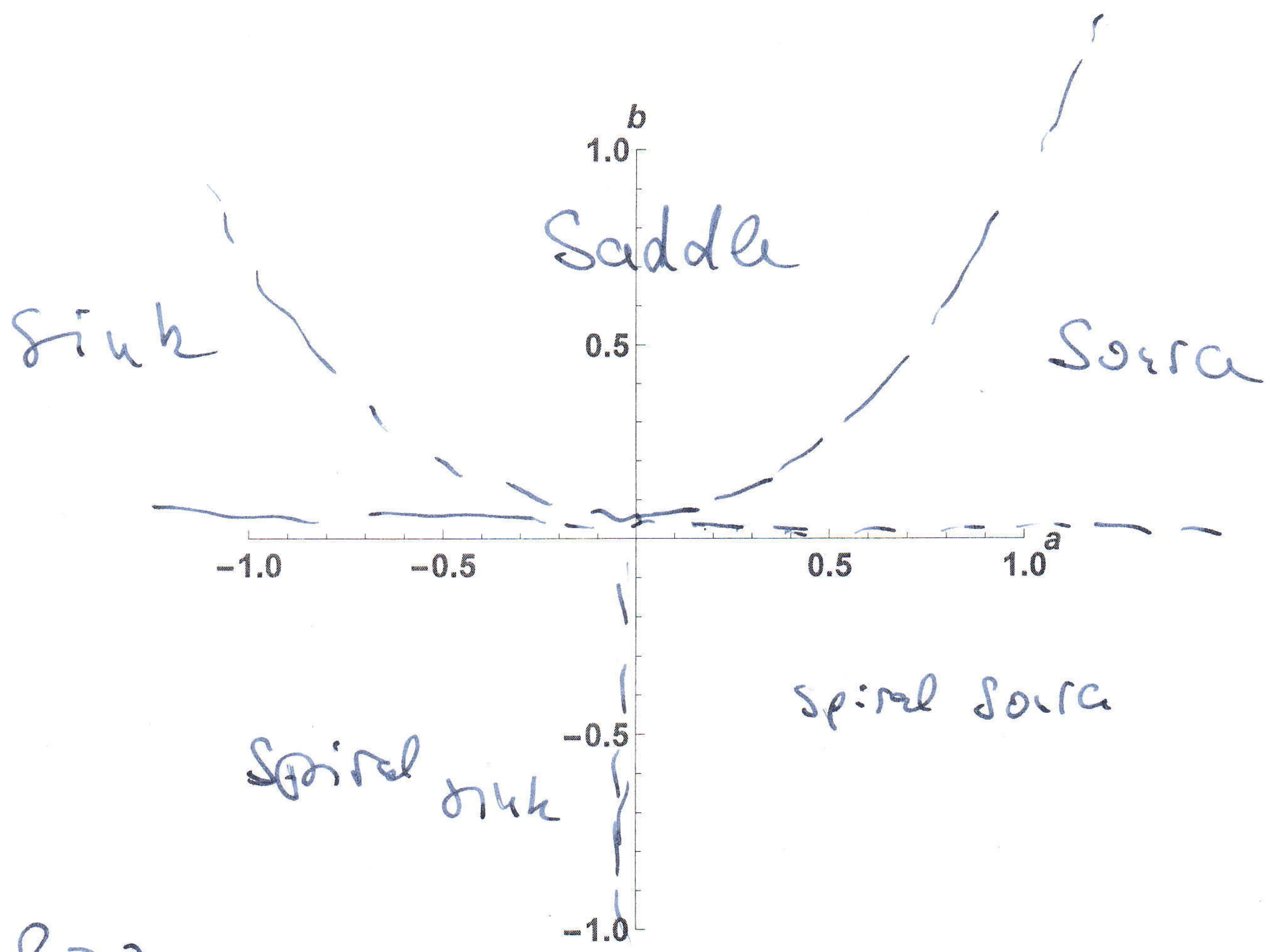
$\underline{b < 0}$ complex eigenv.

$a > 0$ spiral source

$a < 0$ spiral sink

(0 as eigenv.)

$b = a^2$



$\underline{b > 0}$ $b > a^2$ saddle

$b < a^2$, $a > 0$ source

, $a < 0$ sink

Extra Credit Problem 6 (20 points) Suppose two similar countries Y and Z are engaged in an arms race. Let $y(t)$ and $z(t)$ denote the size of the stockpiles of arms of Y and Z, respectively.

The situation is modeled by a system of differential equations of the form

$$y' = f(y, z), \; z' = g(y, z).$$

Suppose the following:

- If country Z's stockpile is not changing, then any increase in size of Y's stockpile results in a decrease in the rate of arms building in country Y. The same is true for country Z.
- If either country increases its stockpile, the other responds by increasing its rate of arms production.

What types of equilibrium points are possible for such a system? Explain carefully!

at the equil pt. $Jac = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The info above implies $a, d < 0$, $b, c > 0$

charact. eq.: $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

with solutions:

$$\lambda = \frac{a+d}{2} \pm \frac{\sqrt{(a+d)^2 - 4ad + 4bc}}{2}$$

$$= \underbrace{\frac{a+d}{2}}_{<0} \pm \underbrace{\frac{\sqrt{(a-d)^2 + 4bc}}{2}}_{>0 \text{ (and real)}}$$

so equil. pt is a sink or a saddle