Problem 3.6. If n is a natural number, then $n! \geq 2^{n-1}$.

Proof by Induction: $n \in \mathbb{N}$

Base Case:

$n = 1, 1! \geq 2^{1-1} \leftrightarrow 1 \geq 1$ TRUE

Induction Step:

$P(k): k! \geq 2^{k-1}$

$P(k+1): (k+1)! \geq 2^{k-1+1} \leftrightarrow (k+1)k! \geq 2^k$
$\leftrightarrow 2^{k-1}(k+1) \geq 2^k$

Show: $(k+1)! \geq 2^k$

$(k+1)! = k!\,(k+1) = 2^{k-1}(k+1) = k2^{k-1} + 2^{k-1}$

Problem 3.7. For all $n \in \mathbb{N}$, the expression $n^2 + n + 41$ is a prime number.

If n=41, then

$P(41): 41^2 + 41 + 41 = 41(41+2) = 41(43) = 1763$

Factors of 1763: 1,41,43,1763

Since 1763 is not a prime number, this makes the statement FALSE.

Statement was proven false through providing a counterexample.

$\forall_n \in \mathbb{N}: n^2 + n + 41$ is a prime number

$\exists_n \in \mathbb{N}: n^2 + n + 41$ is NOT a prime number

Problem 3.9. If n is a natural number, then

$$2 \cdot 6 \cdot 10 \cdot 14 \cdots (4n-2) = \frac{(2n)!}{n!}.$$

Let's say n=1

$$(4(1)-2) = \frac{(2(1))!}{1!} = 2$$

This is a true statement for n=1.

Assume that $2 \cdot 6 \cdot 10 \cdot 14 \cdots (4n-2) = \frac{(2n)!}{n!}$ is true for all n.

Induction: (n+1)

$$2 \cdot 6 \cdot 10 \cdot 14 \cdots (4n-2)(4(n+1)-2)$$

$$= \frac{(2n)!}{n!}(4(n+1)-2)$$

$$= \frac{(2n)!}{n!}(4n+2)$$

$$= \frac{(2n)!}{n!}2(2n+1)$$

$$= \frac{(2n)!2(2n+1)(n+1)}{n!(n+1)}$$

$$= \frac{(2n)!(2n+1)2(n+1)}{(n+1)!}$$

$$= \frac{(2n)!(2n+1)(2n+2)}{(n+1)!}$$

$$= \frac{(2n+2)!}{(n+1)!}$$

$$= \frac{(2(n+1))!}{(n+1)!}$$

(n+1) is true, therefore, $2 \cdot 6 \cdot 10 \cdot 14 \cdots (4n-2) = \frac{(2n)!}{n!}$ is true for all natural numbers.

Problem 3.12. If n is a natural number and $n \geq 4$, then $n! > n^2$. [Note that the inequality is false if $n < 4$.]

$P(n) = n! > n^2, n \geq 4$

Base Case: $P(4)$: $4! > 4^2 \leftrightarrow 24 > 16$

Assume $P(k)$ is true for some number k: $k! > k^2$

Now see that $P(k+1)$ is true

$(k+1)! > (k+1)^2 \leftrightarrow k!\,(k+1) > (k+1)^2 \leftrightarrow k! > k+1, k \geq 4$

For k=4: $4 \cdot 3 \cdot 2 \cdot 1 > 5 \leftrightarrow 24 > 5 \rightarrow$ TRUE

Problem 3.15. Let $a_1 = 1$, $a_2 = 3$, and for $n \geq 2$ let $a_n = a_{n-1} + a_{n-2}$. Show that $a_n < (7/4)^n$ for all natural numbers.

$P(n)$: $a_n < (\frac{7}{4})^n$, $\forall_n \in \mathbb{N}$

Base Cases: (There are two base cases because $a_n = a_{n-1} + a_{n-2}$ is only defined for $n > 2$)

$a_1 = 1 < (\frac{7}{4})^1$ TRUE

$a_2 = 3 < (\frac{7}{4})^2$ TRUE

Induction Step: $P(k) \to P(k+1)$

We know: $a_k < (\frac{7}{4})^k$, for all $k \in \mathbb{N}, k > 2$

$$a_{k+1} < (\frac{7}{4})^{k+1}, k+1 > 3$$

$a_k = a_{k-1} + a_{k-2}$

$a_{k+1} = a_k + a_{k-1} = a_{k-1} + a_{k-2} + a_{k-1}$

$a_{k-1} + a_{k-2} + a_{k-1} < 2(\frac{7}{4})^{k-1} + (\frac{7}{4})^{k-2}$

$a_{k-1} + a_{k-2} + a_{k-1} < 2(\frac{7}{4})^{k-1} + (\frac{7}{4})^{k-1}$

$a_{k-1} + a_{k-2} + a_{k-1} < 3(\frac{7}{4})^{k-1}$

$a_{k-1} + a_{k-2} + a_{k-1} < (\frac{7}{4})^2(\frac{7}{4})^{k-1}$

$a_{k-1} + a_{k-2} + a_{k-1} < (\frac{7}{4})^{k+1} \leftrightarrow a_{k+1} < (\frac{7}{4})^{k+1}$

Thus, $P(n)$ is true $\forall_n \in \mathbb{N}$

Division Algorithm

Given natural numbers $b < a$ there are unique numbers q and r with

$0 \leq r < b$ such that $a = q \cdot b + r$

Proof by Induction:

Base Case: a=1

Then b=1, so a=b+0

Induction Step: For given a and b we know there are q and r with

$a = q \cdot b + r$

$a + 1 = q \cdot b + (r + 1)$

Case 1: If $r + 1 < b$, we are done

Case 2: If $r + 1 = b \rightarrow a + 1 = q \cdot b + b + 0$