Logic and Proof: Supplement

HELMUT KNAUST

Department of Mathematical Sciences
The University of Texas at El Paso
El Paso TX 79968-0514

November 4, 2022

The set of natural numbers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}$. The set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is denoted by $\mathbb{Z}$. $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{R}$ the set of real numbers.

Intervals are defined as follows for real numbers $a < b$:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

$$\text{and } [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Half-open intervals and rays such as $[a, \infty)$ or $(-\infty, b)$ are defined analogously. (Note that $\pm\infty$ are not considered to be real numbers.)

1 Miscellaneous

S 1.1 Show that $\sqrt{5}$ is not a rational number. ($\sqrt{5}$ denotes the positive real number whose square equals 5.)

S 1.2 For which values of $k \in \mathbb{N}$ is $\sqrt{k}$ an irrational number?

S 1.3 Suppose two positive real numbers $a > b$ satisfy $\dfrac{a}{b} = \dfrac{a+b}{a}$. Show that $\dfrac{a}{b}$ is an irrational number.

S 1.4 Prove or disprove: There is an irrational number r such that $r^{\sqrt{2}}$ is rational.

S 1.5 Show that there are infinitely many prime numbers.

S 1.6 There are infinitely many prime numbers of the form $4k + 3$ for some $k \in \mathbb{N}$, such as $3, 7, 11, 19, 23, 31 \ldots$.

It is also true that there are infinitely many prime numbers of the form $4k + 1$ for some $k \in \mathbb{N}$, such as $5, 13, 17, 29, 33 \ldots$. Strangely, this result seems to be harder to prove than the one above.

S 1.7 If every **even** natural number > 2 is the sum of two primes, then every **odd** natural number > 5 is the sum of three primes.

S 1.8 Prove or disprove: If n is a prime number, then $n + 7$ is not prime..

2 Families of Sets

In this section all sets are subsets of a universal set U.

Let I be a non-empty set, and let A_i be a subset of U for each $i \in I$.

$$\mathcal{F} = \{A_i \mid i \in I\} \text{ is called a } \textit{family of sets}.$$

Then union and intersection of a family of sets is defined as follows:

$$\bigcup_{i\in I} A_i = \{x \in U \mid \exists i \in I : x \in A_i\}$$

$$\bigcap_{i\in I} A_i = \{x \in U \mid \forall i \in I : x \in A_i\}$$

Without explicitly mentioning an index set, one can alternatively define:

$$\bigcup_{A\in \mathcal{F}} A = \{x \in U \mid \exists A \in \mathcal{F} : x \in A\}$$

$$\bigcap_{A\in \mathcal{F}} A = \{x \in U \mid \forall A \in \mathcal{F} : x \in A\}$$

Some problems below require the *Archimedean property* of the real numbers: For all real numbers r there is a natural number n with $n > r$. This implies that for all real numbers $r > 0$ there is a natural number n such that $0 < \frac{1}{n} < r$.

S 2.1 Let B be an arbitrary set and let $\mathcal{F}$ be a family of sets.

$$\left(\bigcup_{A\in\mathcal{F}} A\right) \cup B = \bigcup_{A\in\mathcal{F}} (A \cup B)$$

S 2.2 Let B be an arbitrary set and let $\mathcal{F}$ be a family of sets.

$$\left(\bigcup_{A\in\mathcal{F}} A\right) \cap B = \bigcup_{A\in\mathcal{F}} (A \cap B)$$

S 2.3 Let $\mathcal{F}$ be a family of sets.

$$U \setminus \left(\bigcup_{A\in\mathcal{F}} A\right) = \bigcap_{A\in\mathcal{F}} (U \setminus A)$$

S 2.4 Find $\bigcup_{n\in\mathbb{N}} [0, \frac{1}{n})$.

S 2.5 Find $\bigcap_{n\in\mathbb{N}} [0, \frac{1}{n}]$.

S 2.6 Find $\bigcap_{n\in\mathbb{N}} (0, \frac{1}{n})$.

S 2.7 Find $\bigcap_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n})$

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S 2.8 Find $\bigcup_{n\in\mathbb{N}} [-n, n]$.

S 2.9 Find $\bigcap_{q\in\mathbb{Q}} (\mathbb{R} \setminus \{q\})$.

S 2.10 Let $\mathbb{R}^+$ denote the set of positive real numbers. Find

$$\bigcup_{r \in \mathbb{R}^+} \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid (x - r)^2 + y^2 = r^2\}.$$

3 Equivalence Relations and Partitions

S 3.1 Let $\sim$ be an equivalence relation on a set X. Recall that the equivalence class $[x]$ of $x \in X$ is defined as

$$[x] = \{y \in X \mid x \sim y\}.$$

Show:

1. $[x] \neq \emptyset$ for all $x \in X$.
2. $[x] = [y]$ or $[x] \cap [y] = \emptyset$ for all $x, y \in X$.
3. $\bigcup_{x \in X} [x] = X$.

S 3.2 Discuss the validity of the three properties above, if $\sim$ is assumed to be

1. symmetric and transitive, but not reflexive.
2. reflexive and symmetric, but not transitive.
3. reflexive and transitive, but not symmetric.

Given a set $X \neq \emptyset$, we say $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *partition of* X if

1. $A \neq \emptyset$ for all $A \in \mathcal{F}$.
2. $A = B$ or $A \cap B = \emptyset$ for all $A, B \in \mathcal{F}$.
3. $\bigcup_{A \in \mathcal{F}} A = X$.

S 3.1 then states that, given an equivalence relation on a set X, its equivalence classes form a partition of X.

S 3.3 Let $\mathcal{F}$ be a partition of a set X. Show that the relation $\approx$ on X defined by

$$x \approx y \text{ if there is an } A \in \mathcal{F} \text{ such that } x, y \in A$$

is an equivalence relation.

S 3.4 $\mathcal{F} = \{\{1,2,3\},\{4,5\},\{6\}\}$ is a partition on the set $\{1,2,3,4,5,6\}$. Compute $\approx$ as defined in **S** 3.3.

S 3.5 Let $\mathcal{F}$ be the set of equivalence classes of an equivalence relation $\sim$, and let $\approx$ be the equivalence relation induced by $\mathcal{F}$ as defined in **S** 3.3.

Show that $\sim = \approx$.

S 3.6 Let $\mathcal{F}$ be a partition on a set X, and let let $\approx$ be the equivalence relation induced by $\mathcal{F}$ as defined in **S** 3.3. Let $\mathcal{G}$ be the set of equivalence classes of $\approx$.

Show that $\mathcal{F} = \mathcal{G}$.