2   Dedekind Cuts: The Real Numbers

Completeness.   While the rational numbers have nice algebraic properties with respect to their addition, their multiplication and their order, they have one crucial deficiency: The set of rational numbers has "holes".

For instance, the increasing sequence of rational numbers

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

approaches the non-rational number $\sqrt{2}$, a fact well known since antiquity.

We want to remedy this deficiency by constructing an ordered field F containing the rational numbers, which is "complete" in the following sense:

> **(C1)** Every increasing bounded sequence of elements in F converges to
>   an element in F.[14]

Calculus books usually introduce completeness of the set of real numbers in this fashion.

It is convenient to describe completeness also in a different way.

We say a **non-empty** set $A \subseteq F$ is *bounded from above*, if there is a $b \in F$ such that $a \leq b$ for all $a \in A$. Such an element b is then called an upper bound for the set A.

If $A \subseteq F$ is bounded from above, we say that A has a *least upper bound*, denoted by $\sup(A) \in F$, if

1. $\sup(A)$ is an upper bound of A, and

2. for all upper bounds b of A, we have $\sup(A) \leq b$.

Note that $\sup(A)$ must be in F, but we do not require that $\sup(A)$ is an element of A.

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[14]A *sequence* is a function $\phi : \mathbb{N} \to F$.

  A sequence $\phi : \mathbb{N} \to F$ is called *increasing*, if $m \leq n$ implies $\phi(m) \leq \phi(n)$.

  An increasing sequence $\phi : \mathbb{N} \to F$ is called *bounded*, if there is a $b \in F$ such that $\phi(n) \leq b$ for all $n \in \mathbb{N}$.

  We say that the increasing sequence ϕ *converges* to $a \in F$, if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $a - \varepsilon \leq \phi(n) \leq a$ for all $n \geq N$.

Task 2.1
Let $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$. Show that A is bounded from above, but fails to have a least upper bound in $\mathbb{Q}$. This shows that $\mathbb{Q}$ is **not** complete.

The *greatest lower bound* of a set is defined analogously:

We say a non-empty set $A \subseteq F$ is *bounded from below*, if there is a $b \in F$ such that $b \leq a$ for all $a \in A$. Such an element b is then called a lower bound for the set A.

If $A \subseteq F$ is bounded from below, we say that A has a *greatest lower bound*, denoted by $\inf(A) \in F$, if

1. $\inf(A)$ is a lower bound of A, and

2. for all lower bounds b of A, we have $b \leq \inf(A)$.

Task 2.2
Show the following are equivalent:

1. All subsets of F that are bounded from above have a least upper bound.

2. All subsets of F that are bounded from below have a greatest lower bound.

Completeness can then be stated as follows:

(C2) Every subset A of F, which is bounded from above, has a least upper bound.

Task 2.3

Show that property **(C2)** implies property **(C1)**.

Task 2.4

Show that property **(C1)** implies property **(C2)**.

Constructions of the real numbers. Historically, three "constructions" of the real numbers gained prominence in the 19th century, due to RICHARD DEDEKIND (Dedekind cuts), GEORG CANTOR and AUGUSTIN-LOUIS CAUCHY (fundamental sequences), and PAUL BACHMANN (nested intervals), respectively. We will present the first construction below.

Dedekind Cuts. Given two sets of rational numbers $\emptyset \neq L, U \subseteq \mathbb{Q}$, we say that (L, U) is a *partition* of $\mathbb{Q}$ (into two sets), if $L \cup U = \mathbb{Q}$ and $L \cap U = \emptyset$.

A partition (L, U) of $\mathbb{Q}$ is called a *Dedekind cut*, if the following properties hold:

1. If $a \in L$ and $b \in U$, then $a < b$.

2. U has no minimal element.

Here, the element x of a non-empty set A of rational numbers is called *minimal element* of A, if $x \leq a$ for all $a \in A$.

L and U are complementary sets: $U = \mathbb{Q} \setminus L$, and $L = \mathbb{Q} \setminus U$.

We say that two Dedekind cuts (L_1, U_1) and (L_2, U_2) are *equal* and write $(L_1, U_1) = (L_2, U_2)$, if $U_1 = U_2$ (or equivalently, $L_1 = L_2$).

Here are two examples of Dedekind cuts:

> **Task 2.5**
> Show that
> $$L = \{q \in \mathbb{Q} \mid q \leq -3\}, \ U = \{q \in \mathbb{Q} \mid q > -3\}$$
> defines a Dedekind cut.

The two sets above "meet" at the rational number -3.

> **Task 2.6**
> Show that
> $$L = \{q \in \mathbb{Q} \mid q \leq 0 \text{ or } q^2 < 2\}, \ U = \{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 > 2\}$$
> defines a Dedekind cut.

Here the two sets of the Dedekind cut "meet" at the irrational number $\sqrt{2}$.

Dedekind then defined the set of real numbers to be the set of all Dedekind cuts:

$$\mathbb{R} = \{(L, U) \mid (L, U) \text{ is a Dedekind cut}\}.$$

Note that the rational number $q \in \mathbb{Q}$ corresponds to the Dedekind cut, defined by $L = (-\infty, q] \cap \mathbb{Q}$, $U = (q, \infty) \cap \mathbb{Q}$. We will denote this Dedekind cut by $\underline{q}$.

Addition of Dedekind cuts. Given two Dedekind cuts (L_1, U_1) and (L_2, U_2) we define their sum to be the Dedekind cut (X, Y), where

$$Y = \{y \in \mathbb{Q} \mid y = u_1 + u_2 \text{ for some } u_1 \in U_1 \text{ and } u_2 \in U_2\},$$

$$\text{and } X = \mathbb{Q} \setminus Y.$$

Task 2.7

Show that (X, Y) is indeed a Dedekind cut.

Task 2.8

Let $p, q \in \mathbb{Q}$. Show: $\underline{p} + \underline{q} = \underline{p + q}$.

The next three tasks will establish that the set of Dedekind cuts with the addition defined above forms an Abelian group (see p. 12).

Task 2.9

Show that the addition of Dedekind cuts is commutative.

Addition of Dedekind cuts is also associative.

Task 2.10

Show that for every Dedekind cut (L, U)

$$(L, U) + \underline{0} = (L, U).$$

Task 2.11

What is the additive inverse of a Dedekind cut?

Note that the previous task makes it, in particular, possible to define the difference of two Dedekind cuts.

Next, we can define an order on Dedekind cuts: We say that $(L_1, U_1) \leq (L_2, U_2)$, if $L_1 \subseteq L_2$. In particular, (L, U) is non-negative, if $(-\infty, 0] \cap \mathbb{Q} \subseteq L$. We say $(L_1, U_1) < (L_2, U_2)$, if $(L_1, U_1) \leq (L_2, U_2)$ and $(L_1, U_1) \neq (L_2, U_2)$

Clearly $\leq$ is reflexive, anti-symmetric and transitive (why?). The order is also total:

Task 2.12
For any two Dedekind cuts (L_1, U_1) and (L_2, U_2),

$$(L_1, U_1) \leq (L_2, U_2) \text{ or } (L_2, U_2) \leq (L_1, U_1).$$

It is harder to define the multiplication of Dedekind cuts. If both (L_1, U_1) and (L_2, U_2) are non-negative, we define their product (X, Y) by setting

$$Y = \{y \in \mathbb{Q} \mid y = u_1 \cdot u_2 \text{ for some } u_1 \in U_1 \text{ and } u_2 \in U_2\},$$

$$\text{and } X = \mathbb{Q} \setminus Y.$$

Task 2.13
Check that the product defined above is indeed a Dedekind cut.

To define the product of arbitrary Dedekind cuts, one first needs the following result:

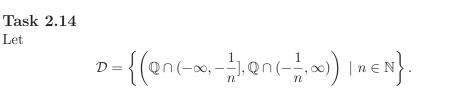**Theorem.** Every Dedekind cut is the difference of two non-negative Dedekind cuts.

The product of two arbitrary Dedekind cuts is then defined by "multiplying out"; the concept is well-defined.

With these definitions one can show with quite a bit more work:

Theorem.   The real numbers with the addition, multiplication and order defined above form an **ordered field**.

The Dedekind cut $\underline{1} := (\mathbb{Q} \cap (-\infty, 1], \mathbb{Q} \cap (1, \infty))$ is the neutral element with respect to multiplication. The existence of a multiplicative inverse is first shown for positive Dedekind cuts, and then generalized to negative Dedekind cuts.

Completeness of Dedekind cuts.   Note that a Dedekind cut (L', U') is an upper bound for a set of Dedekind cuts $\mathcal{D}$, if $L \subseteq L'$ for all $(L, U) \in \mathcal{D}$.

Task 2.14

Let
$$\mathcal{D} = \left\{ \left(\mathbb{Q} \cap (-\infty, -\frac{1}{n}], \mathbb{Q} \cap (-\frac{1}{n}, \infty) \right) \mid n \in \mathbb{N} \right\}.$$

Show that $\mathcal{D}$ is bounded from above, then determine its least upper bound.

Finally we can show that the set of real numbers defined via Dedekind cuts is **complete**:

Task 2.15

Show that $\mathbb{R}$, the set of all Dedekind cuts, satisfies Axiom **(C2)**.

Task 2.16

Show that $\mathbb{Q}$ is dense in $\mathbb{R}$: Given two Dedekind cuts $(L_1, U_1) < (L_2, U_2)$, there is a $q \in \mathbb{Q}$ such that
$$(L_1, U_1) \leq \underline{q} \leq (L_2, U_2).$$