The ordered field of real numbers $\mathbb{R}$ has the **Archimedean Property**:

Task 2.17
The set of natural numbers $\{\underline{n} \mid n \in \mathbb{N}\}$ is not bounded in the set of all Dedekind cuts.

Hint: Use the completeness result above.

Task 2.18
For all **positive** Dedekind cuts (L, U) there is a natural number n such that

$$\underline{1/n} < (L, U).$$

This means, less technically: For all $x \in \mathbb{R}$ with $x > 0$ there is an $n \in \mathbb{N}$ such that

$$\frac{1}{n} < x.$$

Note that the previous two results also hold if ”Dedekind cuts” is replaced by “rational numbers”. This means that the ordered field of rational numbers $\mathbb{Q}$ also has the **Archimedean Property**.

3 Cauchy Sequences and Completion

In this chapter we will construct the set of real numbers in a different fashion, via Cauchy sequences of rational numbers.

As before we consider the set of rational numbers $\mathbb{Q}$ as an ordered field.

We define the *absolute value* of a rational number $x \in \mathbb{Q}$ as follows:

$$|x| = \max\{x, -x\}.$$

Recall that the absolute value on $\mathbb{Q}$ has the following properties:

1. $|x| \geq 0$ with $|x| = 0$ if and only if $x = 0$
2. $|x + y| \leq |x| + |y|$
3. $|x \cdot y| = |x| \cdot |y|$

A sequence (x_n) of rational numbers is called a *Cauchy sequence* if for all $r \in \mathbb{Q}$ with $r > 0$ there is an $N \in \mathbb{N}$ such that for all $m, n \geq N$

$$|x_m - x_n| < r.$$

We will say the sequence (x_n) *converges to* $x \in \mathbb{Q}$ if for all $r \in \mathbb{Q}$ with $r > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|x - x_n| < r.$$

In this case we will write $\lim_{n \to \infty} x_n = x$.

Note that the sequence $1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \ldots$ of rational numbers is a Cauchy sequence that does not converge to a rational number.

Task 3.1
Show that every convergent sequence of rational numbers is a Cauchy sequence.

We will say the two Cauchy sequences (x_n) and (y_n) in $\mathbb{Q}$ are *equivalent* if

$$\lim_{n \to \infty} |x_n - y_n| = 0.$$

Task 3.2
Show that this indeed defines an equivalence relation on the set of rational Cauchy sequences.

As an example, the Cauchy sequence $1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \ldots$ is equivalent to the Cauchy sequence $2,\ 1.5,\ 1.42,\ 1.415,\ 1.4143,\ \ldots$

Similarly the constant sequence $1, 1, 1, 1, \ldots$ is equivalent to the Cauchy sequence $0.9, 0.99., 0.999, \ldots.$

We will denote by $\mathcal{R}$ the set of equivalence classes of Cauchy sequences of rational numbers. Given a Cauchy sequence of rational numbers (x_n), we will denote its equivalence class by $[(x_n)]$.

$\mathcal{R}$ will be a model for the set of real numbers. To see this we first need to define addition, multiplication and an order, and afterwards we will "spot"-check that $\mathcal{R}$ satisfies all the properties of an ordered field with these operations. Finally we will establish that $\mathcal{R}$ is complete.

Every Cauchy sequence is bounded:

Task 3.3
Given a Cauchy sequence (x_n) of rational numbers there is a rational number M such that for all $n \in \mathbb{N}$:

$$|x_n| \leq M.$$

Task 3.4
If (x_n) is a Cauchy sequence of rational numbers, $(|x_n|)$ is also a Cauchy sequence.

Task 3.5
Let us define $[(x_n)] + [(y_n)]$ to be $[(x_n + y_n)]$. Show that this is well defined. You have to show first that the sum of two Cauchy sequences is a Cauchy sequence.

The neutral element of addition is $[(0)]$, the equivalence class of the constant sequence of 0s, the additive inverse of $[(x_n)]$ is $[(-x_n)]$.

Task 3.6
Let us define $[(x_n)] \cdot [(y_n)]$ to be $[(x_n \cdot y_n)]$. Show that this is well defined. You have to show first that the product of two Cauchy sequences is a Cauchy sequence.

The neutral element of multiplication is $[(1)]$, the equivalence class of the constant sequence of 1s.

Given a sequence (x_n) and a sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$, we say the sequence (x_{n_k}) is a *subsequence* of (x_n). For instance, if $n_k = k^2$ for all $k \in \mathbb{N}$, then (x_{n_k}) is the sequence $x_1, x_4, x_9, x_{16}, x_{25}, \ldots$

Task 3.7
Let (x_n) be a sequence of rational numbers. Then any subsequence (x_{n_k}) of (x_n) is a Cauchy sequence and $[(x_n)] = [(x_{n_k})]$.

Task 3.8

1. Given $[(x_n)] \neq [(0)]$, there is a Cauchy sequence (y_n) with $y_n \neq 0$ for all $n \in \mathbb{N}$ such that $[(x_n)] = [(y_n)]$.
2. $(\frac{1}{y_n})$ is a Cauchy sequence and $[(x_n)] \cdot [(\frac{1}{y_n})] = [(1)]$.

The definition of an order on $\mathcal{R}$ is more complicated. We say that $[(x_n)]$ is non-negative if for all $r \in \mathbb{Q}$ with $r > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$

$$x_n > -r.$$

Task 3.9
Using the definition above, show that $[(-\frac{1}{n})]$ is non-negative.

Task 3.10
Show that an equivalence class of Cauchy sequences $[(x_n)]$ is non-negative if and only if there is a $(y_n) \in [(x_n)]$ with $y_n \geq 0$ for all $n \in \mathbb{N}$.

Task 3.11
Show that non-negativeness is well defined.

Finally we say $[(x_n)] \leq [(y_n)]$ if $[(y_n - x_n)]$ is non-negative. Note that $[(0)] \leq [(x_n)]$ if and only if $[(x_n)]$ is non-negative. We write $[(x_n)] > [(0)]$ if $[(0)] \leq [(x_n)]$ and $[(x_n)] \neq [(0)]$.

Clearly this order "$\leq$" is reflexive and transitive.

Task 3.12
Show that the order is anti-symmetric: Given two equivalence classes of Cauchy sequences $[(x_n)]$ and $[(y_n)]$, if $[(x_n)] \leq [(y_n)]$ and $[(y_n)] \leq [(x_n)]$ both hold, then $[(x_n)] = [(y_n)]$.

Task 3.13
Show that the order is total: Given two equivalence classes of Cauchy sequences $[(x_n)]$ and $[(y_n)]$, we have $[(x_n)] \leq [(y_n)]$ or $[(y_n)] \leq [(x_n)]$.

Task 3.14
Show: If $[(x_n)] \leq [(y_n)]$ and $[(z_n)] \in \mathcal{R}$, then $[(x_n)] + [(z_n)] \leq [(y_n)] + [(z_n)]$.

Task 3.15
Show: If $[(x_n)] \leq [(y_n)]$ and $[(z_n)]$ is non-negative, then $[(x_n)] \cdot [(z_n)] \leq [(y_n)] \cdot [(z_n)]$.

Checking all properties required, one obtains that $\mathcal{R}$ is indeed an ordered field.