In the past two weeks, we have explored the many different effects iteration has on linear functions. This paper will go into depth on convergence and divergence, the causes behind the two and the relationship the function has over determining convergence. We have compiled a series of examples, explanations and visual graphs to help us demonstrate our findings.

$a x_n + b = x_{n+1}$

The main topic we will be discussing is the iteration of a linear equation. The formula for a linear equation we will be using is $ax_0 + b = x_n$, where a and b are chosen values and x_0 is the value that will be changing. Iteration in mathematics is the repetition of a process, in this case the repetition of a linear equation. By choosing easy numbers for the initial values we can see how iteration works:

a = 1, b = 1, x_0 = 1
1x_0+1 = x_n, x_0 = 1
When we plug in 1 for x_0 we obtain a new value for x_0
1(1) +1 = 2, x_1= 1
1(2) + 1 = 3, x_2 = 2
1(3) + 1 = 4, x_3 = 3
..
..

As you can see, the solution for the function increases with every new iteration. If one were to continue the iterations forever we would see that the function tends towards infinity. We call this divergence. On the other hand, if the function were to tend towards a number instead of infinity, we would say the function converges. An example of this would be a function with the starting values of:

(there are other possibilities (a = -1, b = 0)

a = 0.5, b = 4, x_0= 4
0.5x_0+ 4 = x_n, x_0 = 4
0.5(4) + 4 = 6, x_1= 4
0.5(6) + 4 = 7, x_2= 6
0.5(7) + 4 = 7.5, x_3= 7
0.5(7.5) + 4 = 7.875, x_4= 7.5
0.5(7.875) + 4 = 7.9375, x_5= 7.875
.
.

3; definition, a div. sequence is a sequence that does not converge; no mention of ∞

From this example we can see that the function is tending towards the number 8, because of this we say that it converges to 8.

In the previous example, we see that when a = 0.5, b = 4 and x_0 = 4 we obtain convergence. However, do we only obtain convergence with those specific values or are there more values that would give us convergence? By using different values for a, b and x_0 we can find patterns and relationships between the numbers and convergence:

Chart 1	[a, b x_0]	Tends towards		[a, b x_0]	Tends towards
A1	[5, 3, 8]	Divergence, ∞	B1	[0.5, 2, -3]	Convergence, 4
A2	[-3, 5, 2]	Divergence, ∞	B2	[0.5, 2, 7]	Convergence, 4
A3	[5.5, 3.1, 53]	Divergence, ∞	B3	[0.5, 2, 9]	Convergence, 4
A4	[0.2, 3, 2]	Convergence, 3.75	B4	[34.43, 2, 78]	Divergence, ∞
A5	[-7, 2, 1]	Divergence, ∞	B5	[7, 2, 456]	Divergence, ∞
A6	[5, 8, -2]	Convergence. -2	B6	[-45, -34, 32]	Divergence, ∞
A7	[45, -2, 7]	Divergence, ∞	B7	[-23, 33, -8]	Divergence, ∞
A8	[0.5, 0.2, 70]	Convergence, 0.4	B8	[0.123, 33, 9]	Convergence, 37.62
A9	[-0.5, -0.2, 80]	Convergence. -0.13	B9	[5, 3, 8]	Divergence, ∞

This table tells us a lot about iterated functions. As you can see, convergence largely occurs when our "a" value is a decimal (A4, A8, A9). It also shows that the function will converge to the same value regardless of x_0,(B1, B2, B3) if a and b stay the same. Additionally, the table shows that when "a" is not a decimal, it appears we obtain divergence for all x_0. We can see through this table that it is possible for a function to converge even if it is not a decimal (A6), but why is this? If we assume that a function does converge, then that means at some point when we plug in the value for x_0 we will obtain x_0:

nice

[0.5, 4, 4]

$0.5x_0 + 4 = x_n,\ x_0 = 4$

$0.5(4) + 4 = 6,\ x_1 = 4$

$0.5(6) + 4 = 7,\ x_2 = 6$

$0.5(7) + 4 = 7.5,\ x_3 = 7$

$$0.5(7.5) + 4 = 7.875,\ x_4 = 7.5$$
$$0.5(7.875) + 4 = 7.9375,\ x_5 = 7.875$$

.
.
.

$$0.5(8) + 4 = 8,\ x_n = 8$$
$$0.5(8) + 4 = 8,\ x_n = 8$$

Is n = 0?

Shown here, when we plug in 8 for the value of x_n, we get 8 back, thus creating a loop and convergence. Knowing this, we can conclude if a function converges, then there must be a point in iteration where the value of x_n will give back the same value of x_n. In other words, at some point in the iteration prosses ax_n + b = x_n will be true. With x_n being the value of convergence. This implies that if not all, then almost all iterated functions converge with some value of x_n.

x_n being equal to $\frac{b}{1-a}$ found by solving for x_n:

$$ax_n + b = x_n \rightarrow b = x_n - ax_n \rightarrow b = x_n(1 - a) \rightarrow x_n = \frac{b}{1-a}$$

If we test this out on our functions we can see that indeed they do converge with a specific value of x_n:

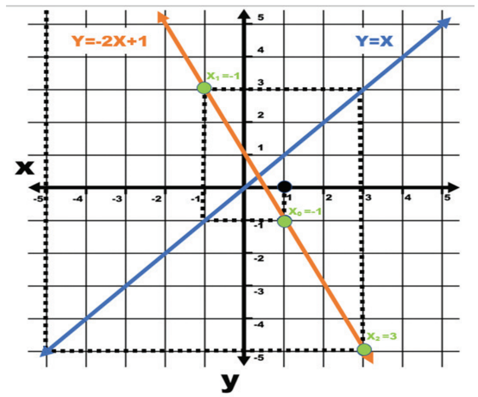
very nice!

Chart 2	[a, b x_0]	$\frac{b}{1-a}$	Tends towards
A1	[5, 3, $\frac{b}{1-a}$]	$\frac{3}{1-5}$= -0.75	Converges, -0.75
A2	[-3, 5, $\frac{b}{1-a}$]	$\frac{5}{1-(-3)}$= 1.25	Converges, 1.25
A3	[5.5, 3.1, $\frac{b}{1-a}$]	$\frac{3.1}{1-5.5}$= -0.688	Converges, -0.688
A5	[-7, 2, $\frac{b}{1-a}$]	$\frac{2}{1-(-7)}$= 0.25	Converges, 0.25

This not only shows us that convergence for every function is possible, but it also shows us that the function will converge to x_0 as long as $x_0 = \frac{b}{1-a}$, giving us the only other way where convergence will not fail. This tells us that for a function to always fail to converge two

1.34 is called "decimal", use $-1 < a < 1$ instead.

conditions must be made: “a” must not be a decimal (-1<a<1), and there can be no a or b value that makes $\frac{b}{1-a}$ equal to anything or simply put “a” cannot equal 1 for $\frac{b}{0}$ is undefined, the only exception being if b = 0 then a must equal a decimal to obtain convergence. We can visualize this conclusion with what is called a cobweb:



So what happens to (x_1) when $a = 1$? What happens to (x_1) if $a = 1$ and $b = 0$?

In this graph, we can visualize the iteration of the function and how it begins to diverge away from the original x_0. With the equation y=-2x+1, the first iteration we can calculate by hand, when $x_0 = 1$, is $x_1 = -1$. When looking back at the graph, we can locate this xvalue by drawing a vertical line from $x_0$to our y=-2x+1 function, then a horizontal line to our y=x linear function and then back up to our original function y=-2x+1. If we continue this process for the next iteration, we will get $x_2 = -3$both by hand and on our graph. By repeating these steps, we can assume the next iteration will give us $x_3 = 5$. In continuing this iteration, we begin to see the cobweb pattern on our graph as it diverges rather than converging to a solution or value.

From everything we have learned thus far, we can see that the functions are divided into different types. Functions like A4, A8, A9 and so on (functions with “a” as a decimal) are Type 1 functions, or functions that give rise to convergent sequences under iteration regardless of the value of x_0. Type 2 functions, which, on iteration, appears to give a divergent sequence for every initial value (A1, A2, A3) but can still converge for some values of x_0. And finally Type 3 functions, functions that never converge.

We now know integrated function can be separated into 3 types as well as that the value of convergence can be determined jointly by a and b through $\frac{b}{1-a}$. But how is the value of convergence, or the limiting value (L), influenced by only one of these variables at a time?

B changing

superfluous: $L = \frac{b}{1-a}$ says it all.

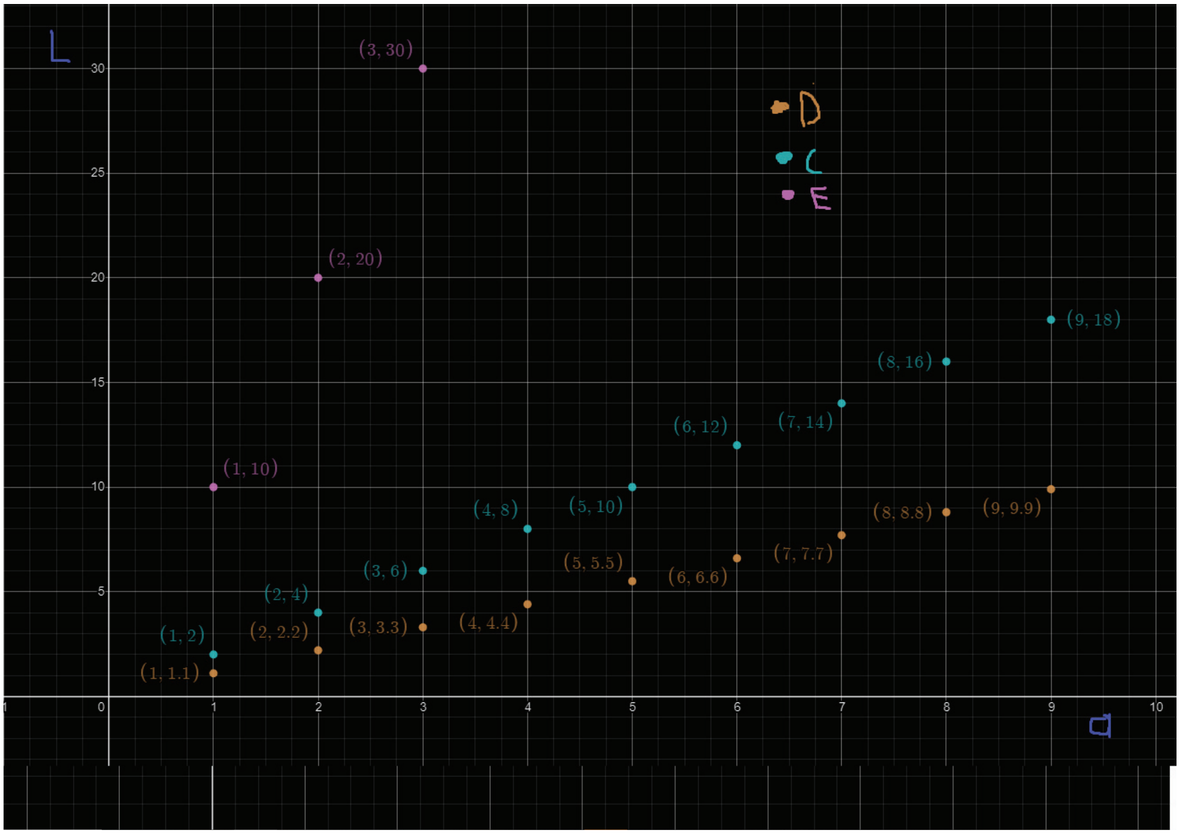
Linear in b,
hyperbolic in a

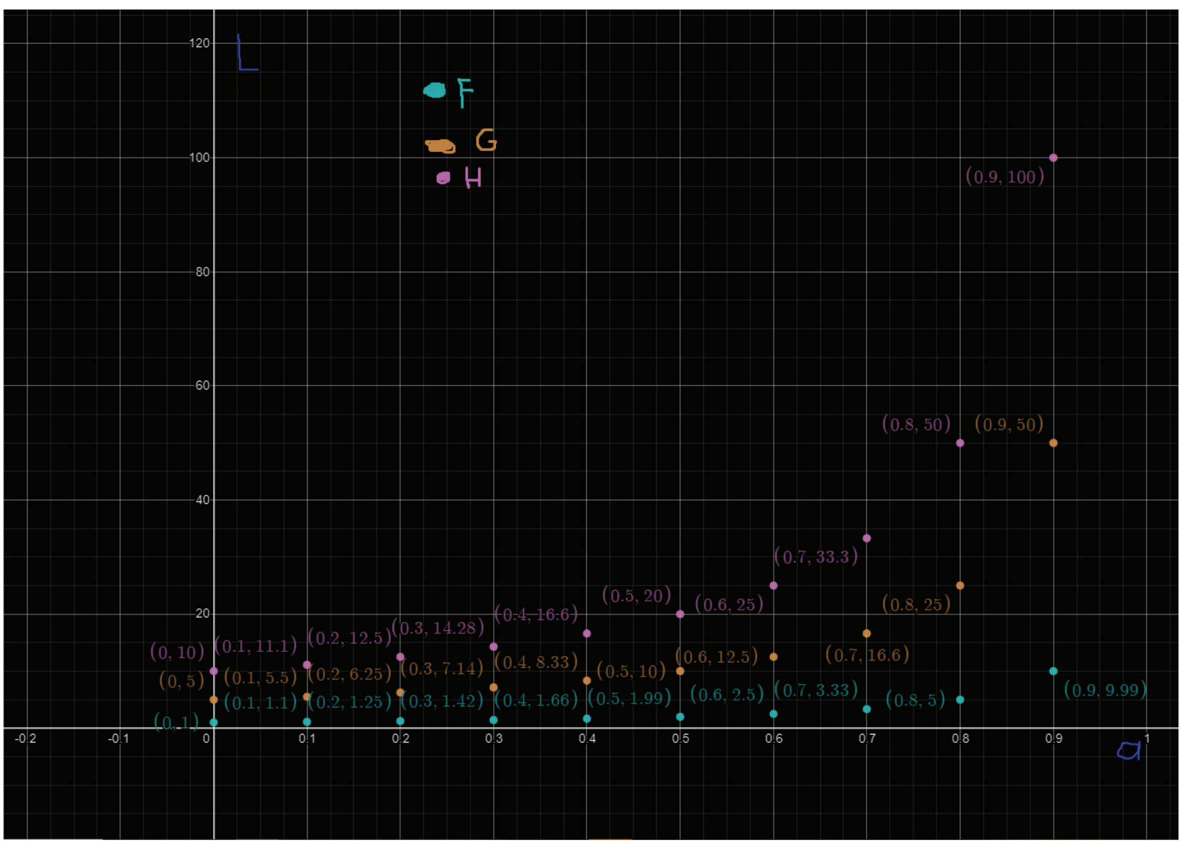
Chart 3	[a, b, x_0, L]		[a, b, x_0, L]		[a, b, x_0, L]
C1	[0.5, 1, 1, 2]	D1	[0.1, 1, 1, 1.1]	E1	[0.9, 1, 1, 9.9]
C2	[0.5, 2, 1, 4]	D2	[0.1, 2, 1, 2.2]	E2	[0.9, 2, 1, 19.9]
C3	[0.5, 3, 1, 6]	D3	[0.1, 3, 1, 3.3]	E3	[0.9, 3, 1, 30]
C4	[0.5, 4, 1, 8]	D4	[0.1, 4, 1, 4.4]	E4	[0.9, 4, 1, 40]
C5	[0.5, 5, 1, 10]	D5	[0.1, 5, 1, 5.5]	E5	[0.9, 5, 1, 50]
C6	[0.5, 6, 1, 12]	D6	[0.1, 6, 1, 6.6]	E6	[0.9, 6, 1, 60]
C7	[0.5, 7, 1, 14]	D7	[0.1, 7, 1, 7.7]	E7	[0.9, 7, 1, 70]
C8	[0.5, 8, 1, 16]	D8	[0.1, 8, 1, 8.8]	E8	[0.9, 8, 1, 80]
C9	[0.5, 9, 1, 18]	D9	[0.1, 9, 1, 9.9]	E9	[0.9, 9, 1, 90]

A changing

Chart 4	[a, b, x_0, L]		[a, b, x_0, L]		[a, b, x_0, L]
F1	[0.1, 1, 1, 1.1]	G1	[0.1, 5, 1, 5.5]	H1	[0.1, 10, 1, 11.1]
F2	[0.2, 1, 1, 1.25]	G2	[0.2, 5, 1, 6.25]	H2	[0.2, 10, 1, 12.5]
F3	[0.3, 1, 1, 1.42]	G3	[0.3, 5, 1, 7.14]	H3	[0.3, 10, 1, 14.28]
F4	[0.4, 1, 1, 1.66]	G4	[0.4, 5, 1, 8.33]	H4	[0.4, 10, 1, 16.6]
F5	[0.5, 1, 1, 1.99]	G5	[0.5, 5, 1, 10]	H5	[0.5, 10, 1, 20]
F6	[0.6, 1, 1, 2.5]	G6	[0.6, 5, 1, 12.5]	H6	[0.6, 10, 1, 25]
F7	[0.7, 1, 1, 3.33]	G7	[0.7, 5, 1, 16.6]	H7	[0.7, 10, 1, 33.3]
F8	[0.8, 1, 1, 5]	G8	[0.8, 5, 1, 25]	H8	[0.8, 10, 1, 50]
F9	[0.9, 1, 1, 9.99]	G9	[0.9, 5, 1, 50]	H9	[0.9, 10, 1, 100]

When reading off tables with a lot of data, it can be hard to fully grasp what the table is telling you. We can do this better by using graphs:


L
a
D
C
E
(3, 30)
(2, 20)
(1, 10)
(9, 18)
(8, 16)
(6, 12)
(7, 14)
(4, 8)
(5, 10)
(3, 6)
(2, 4)
(1, 2)
(8, 8.8)
(9, 9.9)
(7, 7.7)
(5, 5.5)
(6, 6.6)
(2, 2.2)
(3, 3.3)
(4, 4.4)
(1, 1.1)


L
a
F
G
H
(0.9, 100)
(0.8, 50)
(0.9, 50)
(0.7, 33.3)
(0.5, 20)
(0.6, 25)
(0.8, 25)
(0.4, 16.6)
(0.3, 14.28)
(0.2, 12.5)
(0.1, 11.1)
(0, 10)
(0.6, 12.5)
(0.7, 16.6)
(0.5, 10)
(0.4, 8.33)
(0.3, 7.14)
(0.2, 6.25)
(0.1, 5.5)
(0, 5)
(0.1, 1.1)
(0.2, 1.25)
(0.3, 1.42)
(0.4, 1.66)
(0.5, 1.99)
(0.6, 2.5)
(0.7, 3.33)
(0.8, 5)
(0.9, 9.99)
(0, 1)

Once we put all the data in graph form, we can see that the points form a line, in the case where a and x_0 are constant with b changing. It shows L is increasing at a constant rate with respect to b. Additionally we can observe that the rate at which L is increasing, or the slope of the line, is equal to the inverse of a. For example, if we take the middle line, line c, it has a slope of $\frac{2}{1}$. That is, L is increasing at a rate of 2 for every 1 b. If we take the inverse of that slope we get $\frac{1}{2}$, or "a". We can do this process for the other two lines as well and obtain the same conclusion. This tells us, while we are testing the effects of b on L, our chosen "a" will influence our results. This should be expected however, because since L = $\frac{b}{1-a}$. we cannot separate "b" from "a" when talking about L.

Analysis on the second graph, graph J, is harder than the analysis on graph I because the line created by the points increases exponentially. This means as "a" approaches infinity, the rate at which L is increasing also approaches infinity. In addition, as "a" approaches 0 not only does L approach b, but the rate at which it approaches "b" goes to zero.

While analyzing Chart 4 I noticed an interesting ration ship between the L values of different functions that have different b values. The L values for the H column are twice as big as the L values for the G column and ten times as big as the L values for the F column. Looking then at the values of b I then observed the same pattern, the b value for the H column is twice as big as the b value for the G column and ten times as big as the b value for the F column. This can also be observed in Chart 3, the L value of C2 is twice as big as the L value of C1 while the L value of C3 is three times as big as the L values of C1. This same pattern can also be observed with the b values. However, the same pattern cannot be observed for the other two columns, D and E. What I found is that this pattern is just $\frac{b}{1-a}$ in disguise. The reason column H is ten times as big as column F is because the L value for column F is equal to $\frac{1}{1-a}$while the l value of column H is $\frac{10}{1-a}$or the F column times 10. What this shows us is when talking about L we cannot only talk about a or b separately, we must talk about their relationship with L together.

We are close to having an excellent grasp on the iteration of linear functions, however we still must explore the speed of convergence, or how fast a function converges to L. The easiest method to find this would be to manually write down every single step until convergence and then count them. So, this is exactly what we did, by writing some code in the programming language Python that solves the linear function step by step and counts how many steps it took for convergence.

```
w=0
while w != 2:
    h = 0
    inputs = [input(),input(),input()]
    a = inputs[0]
    b = inputs[1]
    x = inputs[2]
    yes = 0
    counter = 0
    while yes != 10000:
        yes = yes + 1
        x=float(a)*float(x)+float(b)
        if h == x:
            break
        counter = counter + 1
        h = x
        print(x)
    print(counter)
```

Very nice!

This code was the easiest and fastest way I found to determine the speed of convergence. With it we can quickly and effectively test which variables most impact the speed of convergence:

s will represent the speed of convergence or how many steps it took to reach convergence

chart 5	[a,b,x_0, s]		[a,b,x_0, s]		[a,b,x_0, s]
K1	[0.1, 6, 6, 16]	L1	[0.5, 1, 6, 54]	M1	[0.5, 5, 1, 53]
K2	[0.2, 6, 6, 23]	L2	[0.5, 2, 6, 52]	M2	[0.5, 5, 2, 53]
K3	[0.3, 6, 6, 30]	L3	[0.5, 3, 6, 1]	M3	[0.5, 5, 3, 53]
K4	[0.4, 6, 6, 40]	L4	[0.5, 4, 6, 52]	M4	[0.5, 5, 4, 53]
K5	[0.5, 6, 6, 53]	L5	[0.5, 5, 6, 52]	M5	[0.5, 5, 5, 52]
K6	[0.6, 6, 6, 71]	L6	[0.5, 6, 6, 53]	M6	[0.5, 5, 6, 52]
K7	[0.7, 6, 6, 99]	L7	[0.5, 7, 6, 53]	M7	[0.5, 5, 7, 52]
K8	[0.8, 6, 6, 159]	L8	[0.5, 8, 6, 53]	M8	[0.5, 5, 8, 51]
K9	[0.9, 6, 6, 331]	L9	[0.5, 9, 6, 53]	M9	[0.5, 5, 9, 50]

*Note that my computer goes to the 15th decimal place, this means that if there is a number with repeating 9's behind it will not return convergence until there are at least 15 9s behind the number. This also means that my computer says the function converges only after 15 decimal places.

Chart 5 shows us not only does a has the greatest impact on the speed of convergence, but it also suggests that b and x_0 have little to no impact on the speed of convergence. Since "b" and x_0 have almost nothing to do with the speed of convergence, there is little to be said about them. We can, however, say multiple things about "a". By looking at the "s" values for the K column and how they compare with each other, we can see that as "a" increases so does "s". Moreover, while "a" increases at a constant rate, we expect "s" to increase exponentially. This means that as "a" approaches one, "s" approaches infinity. This further confirms what we said previously about type 3 functions. Furthermore, as "a" approaches 0 so does s, this is because if a is equal to zero then the function is equal to b and is already convergent, no steps needed to reach convergence.

After experimenting with many different functions and values, we have discovered more about the properties of iteration, what causes convergence and divergence and the impact "a", "b" and x_0 values have on this process. We found that "a" has the greatest impact on almost everything, from determining convergence to the speed at which the function converges. Furthermore, we found that while b has a role to play when determining the value of convergence for Type 2 functions this is almost all it does. Lastly, x_0 does almost nothing.

Summary: Well written, in-depth analysis

For the revision

- Are there cases of type(iii)?
- graph for Question 6 is missing
- Investigate the case $a=1$ more closely