1 Introduction

When we want to study a subject in Mathematics, we first have to agree upon what we assume we all already understand.

In this course we will assume that we are familiar with the Real Numbers, in the sequel denoted by $\mathbb{R}$. Before we list the basic axioms the Real Numbers satisfy, we will briefly review more elementary concepts of numbers.

1.1 The Set of Natural Numbers

When we start learning Mathematics in elementary school, we live in the world of NATURAL NUMBERS, which we will denote by $\mathbb{N}$:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Natural numbers are the "natural" objects to count things around us with. The first thing we learn is to add natural numbers, then later on we start to multiply.

Besides their existence, we will take the following characterization of the Natural Numbers $\mathbb{N}$ for granted throughout the course:

Axiom N1 $1 \in \mathbb{N}$.

Axiom N2 If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Axiom N3 If $n \neq m$, then $n + 1 \neq m + 1$.

Axiom N4 There is no natural number $n \in \mathbb{N}$, such that $n + 1 = 1$.

Axiom N5 If a subset $M \subseteq \mathbb{N}$ satisfies (1) $1 \in M$, and (2) $m \in M \Rightarrow m + 1 \in M$, then $M = \mathbb{N}$.

The first four axioms describe the features of the counting process: We start counting at 1, every counting number has a "successor", and counting is not "cyclic". The last axiom guarantees the **Principle of Induction**:

Task 1.1
Let $P(n)$ be a predicate with domain $\mathbb{N}$. If

1. $P(1)$ is true, and
2. Whenever $P(n)$ is true, then $P(n + 1)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

1.2 Integers, Rational and Irrational Numbers

Deficiencies of the system of natural numbers start to appear when we want to divide—the quotient of two natural numbers is not necessarily a natural number, or when we want to subtract—the difference of two natural numbers is not necessarily a natural number. This leads quite naturally to two extensions of the concept of number.

The set of INTEGERS, denoted by $\mathbb{Z}$, is the set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

The set of RATIONAL NUMBERS $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

Real numbers that are not rational are called IRRATIONAL NUMBERS. The existence of irrational numbers, first discovered by the Pythagoreans in about 520 B.C., must have come as a major surprise to Greek Mathematicians:

Task 1.2
Show that the square root of 2 is irrational. ($\sqrt{2}$ is the positive real number whose square is 2.)

1.3 Groups

Next we will put the properties of numbers and their behavior with respect to the standard arithmetic operations into a wider context by introducing the concept of an "abelian group" and, in the next section, the concept of a "field".

A set G with a binary operation $*$ is called an ABELIAN GROUP[2], if $(G, *)$ satisfies the following axioms:

G1 $*$ is a map from $G \times G$ to G.

G2 (Associativity) For all $a, b, c \in G$

$$(a * b) * c = a * (b * c)$$

G3 (Commutativity) For all $a, b \in G$

$$a * b = b * a$$

[2]Named in honor of Niels Henrik Abel (1802–1829).

G4 (Existence of a neutral element) There is an element $n \in G$, called the neutral element of G, such that for all $a \in G$

$$a * n = a$$

G5 (Existence of inverse elements) For every $a \in G$ there exists $b \in G$, called the inverse of a, such that

$$a * b = n$$

The sets $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are examples of abelian groups when endowed with the usual addition $+$. The neutral element in these cases is 0; it is customary to denote the inverse element of a by $-a$.

The sets $\mathbb{Q} \setminus \{0\} = \{r \in \mathbb{Q} \mid r \neq 0\}$ and $\mathbb{R} \setminus \{0\}$ also form abelian groups under the usual multiplication $\cdot$. In these cases we denote the neutral element by 1; the inverse element of a is customarily denoted by $1/a$ or by a^{-1}.

Exercise 1.3
Write down the axioms **G1–G5** explicitly for the set $\mathbb{Q} \setminus \{0\}$ with the binary operation $\cdot$ (i.e., multiplication).

Addition and multiplication of rational and real numbers interact in a reasonable manner—the following DISTRIBUTIVE LAW holds:

DL For all $a, b, c \in \mathbb{R}$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

1.4 Fields

In short, a set F together with an addition $+$ and a multiplication $\cdot$ is called a FIELD, if

F1 $(F, +)$ is an abelian group (with neutral element 0).

F2 $(F \setminus \{0\}, \cdot)$ is an abelian group (with neutral element 1).

F3 For all $a, b, c \in F$: $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

The set of rational numbers and the set of real numbers are examples of fields.

Another example of a field is the set of complex numbers $\mathbb{C}$:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

Addition and multiplication of complex numbers are defined as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i,$$

respectively.

A field F endowed with a relation $\leq$ is called an ORDERED FIELD if

O1 (Antisymmetry) For all $x, y \in F$

$$x \leq y \text{ and } y \leq x \text{ implies } x = y$$

O2 (Transitivity) For all $x, y, z \in F$

$$x \leq y \text{ and } y \leq z \text{ implies } x \leq z$$

O3 For all $x, y \in F$

$$x \leq y \text{ or } y \leq x$$

O4 For all $x, y, z \in F$

$$x \leq y \text{ implies } x + z \leq y + z$$

O5 For all $x, y \in F$ and all $0 \leq z$

$$x \leq y \text{ implies } x \cdot z \leq y \cdot z$$

If $x \leq y$ and $x \neq y$, we write $x < y$. Instead of $x \leq y$, we also write $y \geq x$.

Both the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ form ordered fields. The complex numbers $\mathbb{C}$ cannot be ordered in such a way.

1.5 The Completeness Axiom

You probably have seen books entitled "Real Analysis" and "Complex Analysis" in the library. There are no books on "Rational Analysis".

Why? What is the main difference between the two ordered fields of $\mathbb{Q}$ and $\mathbb{R}$?—The ordered field $\mathbb{R}$ of real numbers is COMPLETE: sequences of real numbers have the following property.

C Let (a_n) be an increasing sequence of real numbers. If (a_n) is bounded from above, then (a_n) converges.

The ordered field $\mathbb{Q}$ of rational numbers, on the other hand, is **not** complete. It should therefore not surprise you that the Completeness Axiom will play a central part throughout the course! We will discuss this axiom in great detail in Section 2.3.

The complex numbers $\mathbb{C}$ also form a complete field. Section 2.6 will give an idea how to write down an appropriate completeness axiom for the field $\mathbb{C}$.

1.6 Summary: An Axiomatic System for the Set of Real Numbers

Below is a summary of the properties of the real numbers $\mathbb{R}$ we will take for granted throughout the course:

The set of real numbers $\mathbb{R}$ with its natural operations of $+, \cdot$, and $\leq$ forms a complete ordered field. This means that the real numbers satisfy the following axioms:

Axiom 1 $+$ is a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

Axiom 2 For all $a, b, c \in \mathbb{R}$

$$(a + b) + c = a + (b + c)$$

Axiom 3 For all $a, b \in \mathbb{R}$

$$a + b = b + a$$

Axiom 4 There is an element $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$

$$a + 0 = a$$

Axiom 5 For every $a \in \mathbb{R}$ there exists $b \in \mathbb{R}$ such that

$$a + b = 0$$

Axiom 6 $\cdot$ is a map from $\mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\}$ to $\mathbb{R} \setminus \{0\}$.

Axiom 7 For all $a, b, c \in \mathbb{R} \setminus \{0\}$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Axiom 8 For all $a, b \in \mathbb{R} \setminus \{0\}$

$$a \cdot b = b \cdot a$$

Axiom 9 There is an element $1 \in \mathbb{R} \setminus \{0\}$ such that for all $a \in \mathbb{R} \setminus \{0\}$

$$a \cdot 1 = a$$

Axiom 10 For every $a \in \mathbb{R} \setminus \{0\}$ there exists $b \in \mathbb{R} \setminus \{0\}$ such that

$$a \cdot b = 1$$

Axiom 11 For all $a, b, c \in \mathbb{R}$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

Axiom 12 For all $a, b \in \mathbb{R}$

$$a \leq b \text{ and } b \leq a \text{ implies } a = b$$

Axiom 13 For all $a, b, c \in \mathbb{R}$

$$a \leq b \text{ and } b \leq c \text{ implies } a \leq c$$

Axiom 14 For all $a, b \in \mathbb{R}$

$$a \leq b \text{ or } a \geq b$$

Axiom 15 For all $a, b, c \in \mathbb{R}$

$$a \leq b \text{ implies } a + c \leq b + c$$

Axiom 16 For all $a, b \in \mathbb{R}$ and all $c \geq 0$

$$a \leq b \text{ implies } a \cdot c \leq b \cdot c$$

Axiom 17 Let (a_n) be an increasing sequence of real numbers. If (a_n) is bounded from above, then (a_n) converges.

1.7 Maximum and Minimum

Given a non-empty set A of real numbers, a real number b is called MAXIMUM OF THE SET A, if $b \in A$ and $b \geq a$ for all $a \in A$. Similarly, a real number s is called MINIMUM OF THE SET A, if $s \in A$ and $s \leq a$ for all $a \in A$. We write $b = \max A$, and $s = \min A$.

For example, the set $\{1, 3, 2, 0, -7, \pi\}$ has minimum -7 and maximum π, the set of natural numbers $\mathbb{N}$ has 1 as its minimum, but fails to have a maximum.

Exercise 1.4
Show that a set can have at most one minimum.

Task 1.5
Show that finite non-empty sets of real numbers always have a maximum.

Exercise 1.6
Characterize all subsets A of the set of real numbers with the property that $\min A = \max A$.

1.8 The Absolute Value

The ABSOLUTE VALUE of a real number a is defined as

$$|a| = \max\{a, -a\}.$$

For instance, $|4| = 4$, $|-\pi| = \pi$. Note that the inequalities $a \leq |a|$ and $-a \leq |a|$ hold for all real numbers a.

The quantity $|a-b|$ measures the distance between the two real numbers a and b on the real number line; in particular $|a|$ measures the distance of a from 0.

The following result is known as the **triangle inequality**:

Exercise 1.7
For all $a, b \in \mathbb{R}$:

$$|a+b| \leq |a| + |b|$$

A related result is called the **reverse triangle inequality**:

Exercise 1.8
For all $a, b \in \mathbb{R}$:

$$|a-b| \geq \Big| |a| - |b| \Big|$$

You will use both of these inequalities frequently throughout the course.

1.9 Natural Numbers and Dense Sets inside the Real Numbers

In the sequel, we will also assume the following axiom for the Natural Numbers, even though it can be deduced from the Completeness Axiom of the Real Numbers (see Optional Task 2.1):

Axiom N6 For every positive real number $s \in \mathbb{R}$, $s > 0$, there is a natural number $n \in \mathbb{N}$ such that $n - 1 \leq s < n$.

Exercise 1.9
Show that for every positive real number r, there is a natural number n, such that $0 < \frac{1}{n} < r$.

We say that a set A of real numbers is DENSE in $\mathbb{R}$, if for all real numbers $x < y$ there is an element $a \in A$ satisfying $x < a < y$.

Task 1.10
The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Task 1.11
The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.