# 5 The Derivative

## 5.1 Definition and Examples

Let $D$ be a set of real numbers and let $x_0 \in D$ be an accumulation point of $D$. The function $f : D \to \mathbb{R}$ is said to be DIFFERENTIABLE at $x_0$, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

In this case, we call the limit above the DERIVATIVE of $f$ at $x_0$ and write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

**Exercise 5.1**

Use the definition above to show that $\sqrt[3]{x} : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 = -27$ and that its derivative at $x_0 = -27$ equals $\frac{1}{27}$.

**Exercise 5.2**

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x_0 = 0$? See Figure 3 on page 27.

**Exercise 5.3**

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x_0 = 0$? Using your Calculus knowledge, compute the derivative at points $x_0 \neq 0$. Is the derivative continuous at $x_0 = 0$? See Figure 9 on page 44.

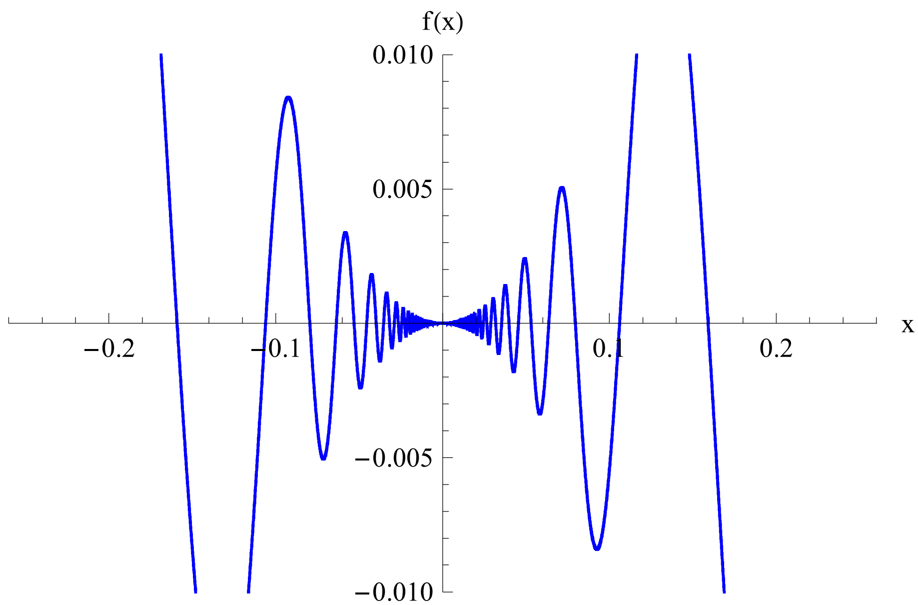


Figure 9: The graph of $x^2 \sin(1/x)$

## 5.2 Techniques of Differentiation

**Exercise 5.4**
Suppose $f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$. Show that $f$ is continuous at $x_0$.

**Exercise 5.5**
Give an example of a function with a point at which $f$ is continuous, but not differentiable.

**Exercise 5.6**
Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f+g$ is differentiable at $x_0$, with $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

Next come some of the "Calculus Classics", beginning with the "Product Rule":

**Task 5.7**
Let $f, g : D \to \mathbb{R}$ be differentiable at $x_0 \in D$. Then the function $f \cdot g$ is differentiable at $x_0$, with

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

In particular, if $c \in \mathbb{R}$, then

$$(c \cdot f)'(x_0) = c \cdot f'(x_0).$$

**Exercise 5.8**
Show that polynomials are differentiable everywhere.

Compute the derivative of a polynomial of the form

$$P(x) = \sum_{k=0}^{n} a_k x^k.$$

***Optional Task 5.1***
State and prove the "Quotient Rule".

***Optional Task 5.2***
State and prove the "Chain Rule".

## 5.3 The Mean-Value Theorem and its Applications

Let $D$ be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function. We say that $f$ has a LOCAL MAXIMUM at $x_0 \in D$, if there is a neighborhood $U$ of $x_0$, such that

$$f(x) \leq f(x_0) \text{ for all } x \in U.$$

Similarly, we say that $f$ has a LOCAL MINIMUM at $x_0 \in D$, if there is a neighborhood $U$ of $x_0$, such that

$$f(x) \geq f(x_0) \text{ for all } x \in U.$$

The next result is commonly known as the **First Derivative Test**. Note that this only works for $x_0 \in (a,b)$, not if $x_0$ is one of the endpoints $a$ or $b$.

### Task 5.9

Suppose $f : [a,b] \to \mathbb{R}$ has either a local maximum or a local minimum at $x_0 \in (a,b)$. If $f$ is differentiable at $x_0$, then $f'(x_0) = 0$.

### Task 5.10

Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on $(a,b)$.

If $f(a) = f(b) = 0$, then there exists a $c \in (a,b)$ with $f'(c) = 0$.

This result is usually called **Rolle's Theorem**, named after Michel Rolle (1652–1719). A much more useful version of Task 5.10 is known as the **Mean Value Theorem**:

### Task 5.11

Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on $(a,b)$.

Then there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

See Figure 10 on page 47.

Do not confuse the Mean Value Theorem with the Intermediate Value Theorem!

Nearly all properties of differentiable functions follow from the Mean Value Theorem. The exercises and tasks below are such examples of straightforward applications of the Mean-Value Theorem.

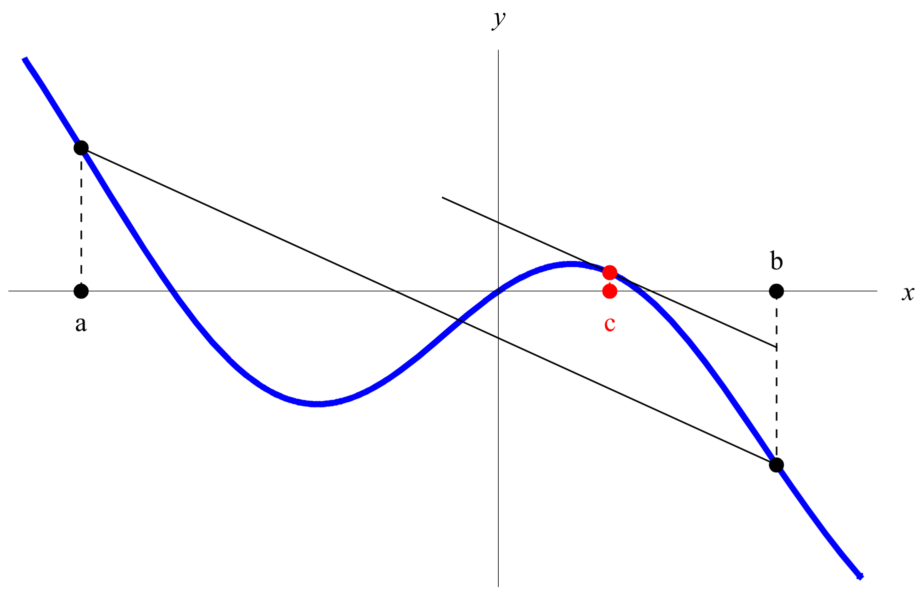


Figure 10: The Mean Value Theorem

**Exercise 5.12**
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$.

If $f'(x) > 0$ for all $x \in (a, b)$, then $f$ is strictly increasing.

**Exercise 5.13**
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$.

If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant on $[a, b]$.

A function $f : D \to \mathbb{R}$ is called INJECTIVE (or 1–1), if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in D$.

**Exercise 5.14**
Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$.

If $f'(x) \neq 0$ for all $x \in (a,b)$, then $f$ is injective.

**Task 5.15**
Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ such that $f'(x) \neq 0$ for all $x \in [a,b]$.

Then $f$ is injective; its inverse $f^{-1}$ is differentiable on $f([a,b])$. Moreover, setting $y = f(x)$, we have

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}.$$

## 5.4 The Derivative and the Intermediate Value Property*

We say that a function $f : [a,b] \to \mathbb{R}$ has the Intermediate Value Property on $[a,b]$ if the following holds: Let $x_1, x_2 \in [a,b]$, and let

$$y \in (f(x_1), f(x_2)).$$

Then there is an $x \in (x_1, x_2)$ satisfying $f(x) = y$.

Recall that we saw earlier that every continuous function has the intermediate value property, see Task 4.18.

On the other hand, not every function with the intermediate value property is continuous:

***Optional Task 5.3***
Let $f : [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0,\ x \in \mathbb{R} \\ 0, & \text{if } x = 0 \end{cases}$$

Show that $f$ has the intermediate value property on the interval $[-1,1]$. See Figure 5 on page 30.

The rest of this section will establish the surprising fact that derivatives have the intermediate value property, even though they are not necessarily continuous (see Task 5.3).

***Optional Task 5.4***
Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$.

If $f'(x) \neq 0$ for all $x \in (a, b)$, then either $f'(x) \geq 0$ for all $x \in [a, b]$ or $f'(x) \leq 0$ for all $x \in [a, b]$.

***Optional Task 5.5***
Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Then $f' : [a, b] \to \mathbb{R}$ has the intermediate value property on $[a, b]$.

## 5.5 A Continuous, Nowhere Differentiable Function*

This section follows the construction in [12]. Another example can be found in [14].

Recall that the LARGEST INTEGER FUNCTION $[x] : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$[x] = k, \text{ if } k \in \mathbb{Z} \text{ satisfies } k \leq x < k + 1.$$

For instance, $[4.5] = 4$ and $[-\pi] = -4$.

We start by defining a 1-periodic function $f_0 : \mathbb{R} \to \mathbb{R}$ as follows[11]:

$$f_0(x) = \begin{cases} x & , \text{if } 0 \leq x - [x] < \frac{1}{2} \\ 1 - x & , \text{if } \frac{1}{2} < x - [x] < 1 \end{cases}$$

See Figure 11 on page 50.

For $n \in \mathbb{N}$, we define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = 2^{-n} f_0(2^n x).$$

Figure 12 on page 51 depicts the function $f_2(x)$.

Finally we let $g_n : [0, 1] \to \mathbb{R}$ for $n \in \mathbb{N}$ be defined as

$$g_n(x) = \sum_{k=0}^{n} f_k(x),$$

[11] A function $f : \mathbb{R} \to \mathbb{R}$ is called $p$-periodic if $f(x + p) = f(x)$ for all $x \in \mathbb{R}$.

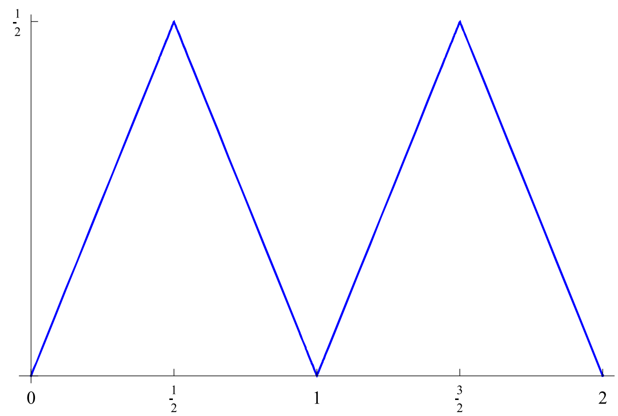


Figure 11: The function $f_0(x)$

and then set

$$g(x) = \lim_{n\to\infty} g_n(x)$$

for all $x \in [0,1]$. Figure 13 on page 52 shows the function $g(x)$.

The function $g(x)$ is continuous on the interval $[0,1]$, but fails to be differentiable at all points in the interval $(0,1)$. To establish these properties we start with

***Optional Task 5.6***

1. For $n \in \mathbb{N} \cup \{0\}$, the function $f_n(x)$ is continuous on $[0,1]$ and $2^{-n}$-periodic.
2. For $n \in \mathbb{N} \cup \{0\}$, the function $f_n(x)$ satisfies $0 \leq f_n(x) \leq 2^{-(n+1)}$ for all $x \in [0,1]$.
3. Show that the estimate

   $$|g_m(x) - g_n(x)| \leq 2^{-(1+\min\{m,n\})}$$

   holds for all $x \in [0,1]$ and all $m, n \in \mathbb{N} \cup \{0\}$.
4. Show that $g(x)$ is well-defined for all $x \in [0,1]$.
5. The function $g(x)$ maps the interval $[0,1]$ into itself.

Using the results above, show:

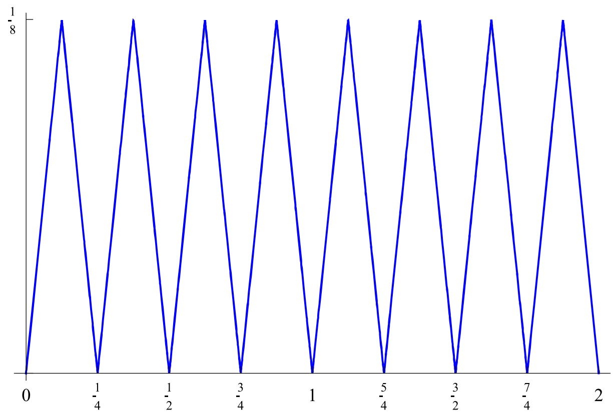


Figure 12: The function $f_2(x)$

***Optional Task 5.7***
The function $g(x)$ is continuous on $[0,1]$.

We will now establish that the function $g(x)$ is nowhere differentiable. First we need the following result:

***Optional Task 5.8***
Let a function $f : [0,1] \to \mathbb{R}$ be differentiable at the point $y \in (0,1)$. Then

$$\lim \frac{f(z) - f(x)}{z - x} \text{ exists and equals } f'(y).$$

Here the limit is taken over all $x, z \in [0,1]$ satisfying $x \leq y \leq z$ and $x \neq y$ such that $\max\{|y-x|, |z-y|\} \to 0$.

More precisely this means the following: For all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \frac{f(z) - f(x)}{z - x} - f'(y) \right| < \varepsilon$$

for all $x, z \in [0,1]$ satisfying $x \leq y \leq z$, $x \neq y$, $|y - x| < \delta$ and $|z - y| < \delta$.

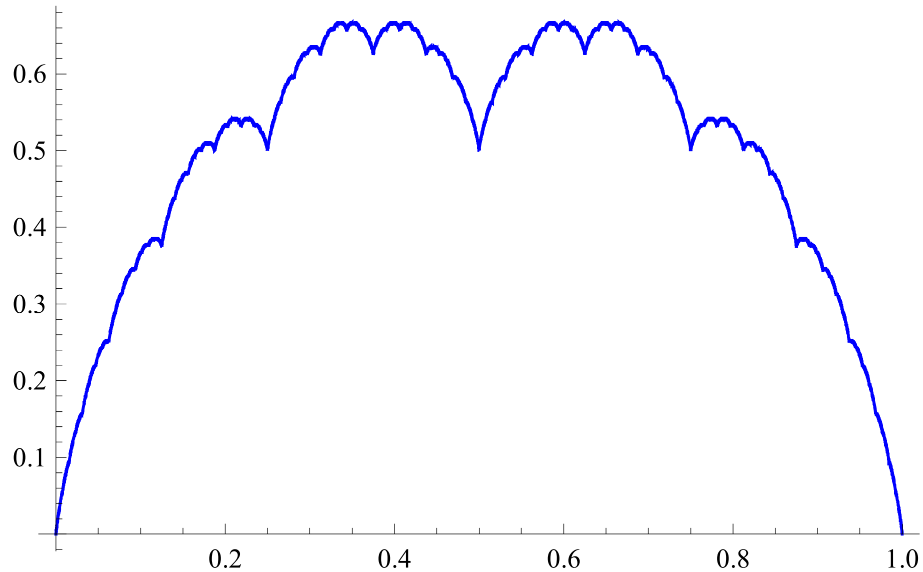


Figure 13: The function $g(x)$

The crucial step is the next task:

***Optional Task 5.9***

For all $y \in (0,1)$ there are four sequences $(x_n)$, $(x'_n)$, $(z_n)$, and $(z'_n)$ in $[0,1]$ with the following properties:

1. All four sequences converge to $y$.
2. $x_n \le y \le z_n$, $x_n \neq z_n$ for all $n \in \mathbb{N}$.
3. $x'_n \le y \le z'_n$, $x'_n \neq z'_n$ for all $n \in \mathbb{N}$.
4. $\left| \dfrac{g(z_n) - g(x_n)}{z_n - x_n} - \dfrac{g(z'_n) - g(x'_n)}{z'_n - x'_n} \right| \ge 1$ for all $n \in \mathbb{N}$.

The proof is somewhat technical. Let $p \in \mathbb{N}$ be such that $\dfrac{p}{2^n} \le y < \dfrac{p+1}{2^n}$. Then choose $x_n$, $z_n$, $x'_n$ and $z'_n$ suitably from the set $\left\{ \dfrac{p}{2^n}, \dfrac{2p+1}{2^{n+1}}, \dfrac{p+1}{2^n} \right\}$. Figure 14 on page 54 shows a typical scenario (for $n = 11$ and $p = 172$).

Finally one can use the last two tasks to show:

***Optional Task 5.10***
The function $g : [0,1] \to [0,1]$ fails to be differentiable at all points in $(0,1)$.

Since $g(x)$ is continuous on the interval $[0,1]$, it has a maximum.

***Optional Task 5.11***
Show that the maximal value of $g(x)$ on the interval $[0,1]$ is $\frac{2}{3}$.

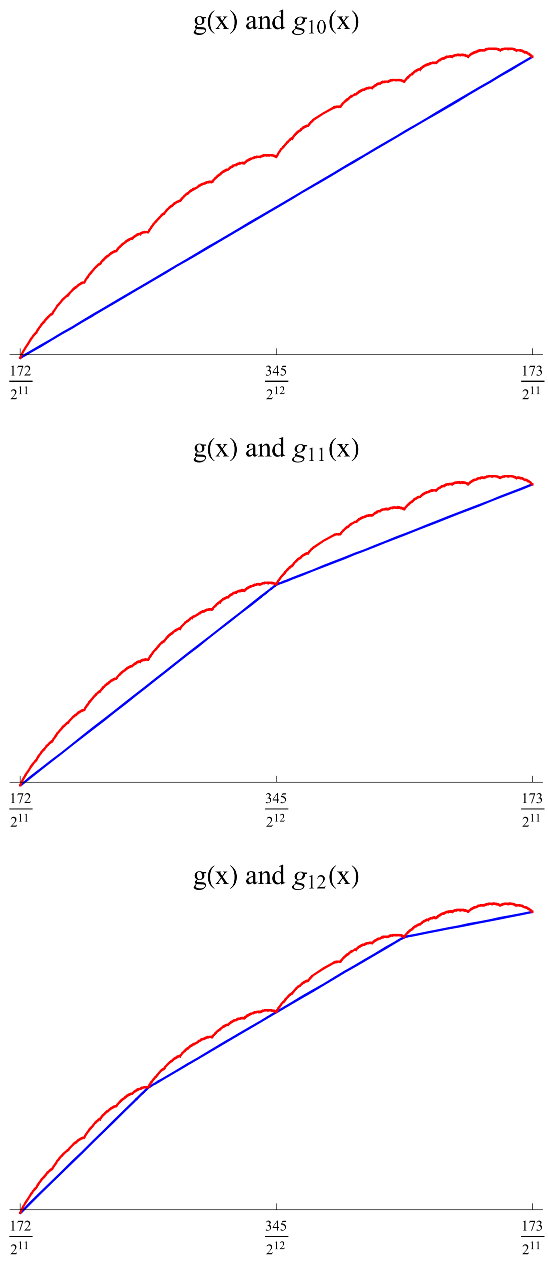


Figure 14: The pictures show the functions $g(x)$ and $g_{10}(x)$, $g(x)$ and $g_{11}(x)$, and $g(x)$ and $g_{12}(x)$, respectively.