6 The Integral

Throughout this chapter all functions are assumed to be bounded.

6.1 Definition and Examples

A finite set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers is called a PARTITION of the interval $[a, b]$, if

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

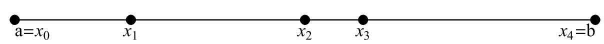

Figure 15: A 5-element partition of the interval $[a, b]$.

Let a function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of the interval $[a, b]$ be given. Let $i \in \{1, 2, 3, \ldots, n\}$. We define

$$m_i(f) := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

and

$$M_i(f) := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

The LOWER RIEMANN SUM $\mathcal{L}(f, P)$ of the function f with respect to the partition P is defined as

$$\mathcal{L}(f, P) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}).$$

See Figure 16 on page 56.

Analogously, the UPPER RIEMANN SUM $\mathcal{U}(f, P)$ is defined as

$$\mathcal{U}(f, P) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

See Figure 17 on page 57.

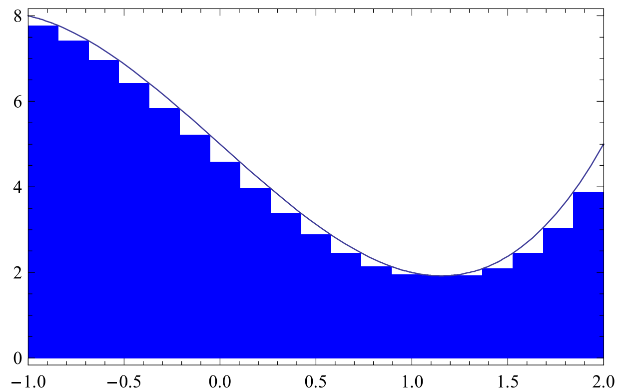

Figure 16: A lower Riemann sum with a partition of 20 equally spaced points.

The LOWER RIEMANN INTEGRAL of f on the interval $[a, b]$, denoted by $\mathcal{L}\int_a^b f(x)\,dx$, is defined as

$$\mathcal{L}\int_a^b f(x)\,dx := \sup\{\mathcal{L}(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Similarly, the UPPER RIEMANN INTEGRAL of f on the interval $[a, b]$ is defined as

$$\mathcal{U}\int_a^b f(x)\,dx := \inf\{\mathcal{U}(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

Let P and Q be two partitions of the interval $[a, b]$. We say that the partition Q is FINER than the partition P if $P \subseteq Q$. In this situation, we also call P COARSER than Q.

Task 6.1
Let $f : [a, b] \to \mathbb{R}$ be a function, and P and Q be two partitions of the interval $[a, b]$. Assume that Q is finer than P. Then

$$\mathcal{L}(f, P) \leq \mathcal{L}(f, Q) \leq \mathcal{U}(f, Q) \leq \mathcal{U}(f, P).$$

Note that Task 6.1 implies that

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

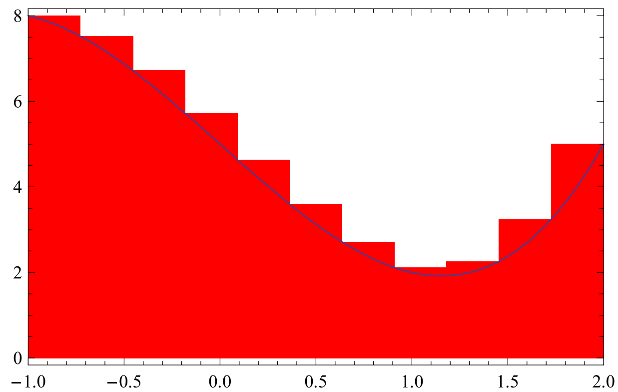

Figure 17: An upper Riemann sum with a partition of 12 equally spaced points.

We are finally in a position to define the concept of integrability! We will say that a function $f : [a, b] \to \mathbb{R}$ is RIEMANN INTEGRABLE on the interval $[a, b]$, if

$$\mathcal{L}\int_a^b f(x)\,dx = \mathcal{U}\int_a^b f(x)\,dx.$$

Their common value is then called the RIEMANN INTEGRAL of f on the interval $[a, b]$ and denoted by

$$\int_a^b f(x)\,dx.$$

Exercise 6.2

Use the definition above to compute $\mathcal{L}\int_0^1 x\,dx$ and $\mathcal{U}\int_0^1 x\,dx$. Is the function Riemann integrable on $[0, 1]$?

Exercise 6.3

Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Use the definitions above to compute $\mathcal{L}\int_0^1 f(x)\,dx$ and $\mathcal{U}\int_0^1 f(x)\,dx$. Is the function Riemann integrable on $[0, 1]$?

Task 6.4
A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) < \varepsilon.$$

The following "Lemma" is quite technical; it prepares for the proof of the subsequent result. Note that given two partitions P and Q, the partition $P \cup Q$ is finer than both P and Q.

Given a partition P, we define its MESH WIDTH $\mu(P)$ as

$$\mu(P) := \max\{x_i - x_{i-1} \mid i = 1, 2, \ldots, n\}.$$

Task 6.5
Let $f : [a,b] \to \mathbb{R}$ be a bounded function with $|f(x)| \leq M$ for all $x \in [a,b]$. Let $\varepsilon > 0$ be given, and let P_0 be a partition of $[a,b]$ with $n+1$ elements. Then there is a $\delta > 0$ (depending on M, n and ε) such that for any partition P of $[a,b]$ with mesh width $\mu(P) < \delta$

$$\mathcal{U}(P) < \mathcal{U}(P \cup P_0) + \varepsilon \qquad \text{and} \qquad \mathcal{L}(P) > \mathcal{L}(P \cup P_0) - \varepsilon.$$

Task 6.6
A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for **all** partitions P of $[a,b]$ with mesh width $\mu(P) < \delta$

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) < \varepsilon.$$

Two important classes of functions are Riemann-integrable—continuous functions and monotone functions:

Task 6.7
If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$.

Task 6.8
If $f : [a, b] \to \mathbb{R}$ is increasing on $[a, b]$, then f is Riemann integrable on $[a, b]$.

An analogous result holds for decreasing functions, of course.

6.2 Arithmetic of Integrals

Exercise 6.9
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Then for all $\lambda \in \mathbb{R}$, the function λf is also Riemann integrable on $[a, b]$, and

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx.$$

Exercise 6.10
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Then $f + g$ is also Riemann integrable on $[a, b]$, and

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Task 6.11
Let $f : [a, c] \to \mathbb{R}$ be a function and $a < b < c$. Then f is Riemann integrable on $[a, c]$ if and only if f is Riemann integrable on both $[a, b]$ and $[b, c]$. In this case

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Exercise 6.12
Suppose the function $f : [a, b] \to \mathbb{R}$ is bounded above by $M \in \mathbb{R}$: $f(x) \le M$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)\,dx \le M \cdot (b - a).$$

6.3 The Fundamental Theorem of Calculus

Task 6.13
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then there is a number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Task 6.14
Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable on $[a, b]$. Let

$$F(x) = \int_a^x f(\tau)\,d\tau.$$

Then $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$.

Does your proof of the result above actually yield a stronger result?

Task 6.15
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Let

$$F(x) = \int_a^x f(\tau)\,d\tau.$$

Then $F : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, and

$$F'(x) = f(x).$$

The next result is the **Fundamental Theorem of Calculus**.

Task 6.16

Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and suppose $F : [a, b] \to \mathbb{R}$ is an "anti-derivative" of $f(x)$, i.e., F satisfies:

1. F is continuous on $[a, b]$ and differentiable on (a, b),
2. $F'(x) = f(x)$ for all $x \in [a, b]$.

Then

$$\int_a^b f(\tau)\, d\tau = F(b) - F(a).$$