2 The Integers

Definition. Integers can be written as differences of natural numbers. The set of integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ will therefore be defined as certain equivalence classes of the two-fold Cartesian product of $\mathbb{N}$.

We define a relation on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c.$$

The next three tasks show that "$\sim$" defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$:

Task 2.1
"$\sim$" is reflexive.

Task 2.2
"$\sim$" is symmetric[12].

Task 2.3
"$\sim$" is transitive.

We will denote equivalence classes as follows:

$$(a, b)_\sim := \{(c, d) \mid (c, d) \sim (a, b)\}.$$

The set of integers $\mathbb{Z}$ is the set of all equivalence classes obtained in this manner:

$$\mathbb{Z} = \{(a, b)_\sim \mid a, b \in \mathbb{N}\}.$$

[12]A relation $\sim$ on A is called *symmetric*, if for all $a, b \in A$ the following holds: $a \sim b$ implies $b \sim a$.

Addition of integers will be defined component-wise:

$$(a,b)_\sim + (c,d)_\sim = (a+c, b+d)_\sim.$$

A set G with a binary operation $\star$ is called an *Abelian group* if $\star$ is commutative and associative, if $(A,\star)$ has a neutral element n satisfying $g \star n = g$ for all $g \in G$, and if $(A,\star)$ has inverse elements, i.e., for all $g \in G$ there is an $h \in G$ satisfying $g \star h = n$.

The next five tasks will show that $\mathbb{Z}$ is an Abelian group with respect to addition.

Task 2.4

Show that the addition of integers is well-defined (i.e. independent of the chosen representatives of the equivalence classes).

Task 2.5

Show that the addition of integers is commutative.

Task 2.6

Show that the addition of integers is associative.

Task 2.7

Show that the addition of integers has $(0,0)_\sim$ as its neutral element.

Task 2.8

Show that for all $a, b \in \mathbb{N}$ the following holds: $(a,b)_\sim + (b,a)_\sim = (0,0)_\sim$. Thus every element in $\mathbb{Z}$ has an additive inverse element.

Task 2.9

1. The map $\phi : \mathbb{N} \to \mathbb{Z}$ defined by $\phi(n) = (n, 0)_\sim$ is injective.
2. For all $m, n \in \mathbb{N}$ the following holds: $\phi(m) + \phi(n) = \phi(m + n)$.

From now on we will **identify** $\mathbb{N}$ with $\phi(\mathbb{N})$. For instance for $(k, 0)_\sim$, we write $k = (k, 0)_\sim$, and consequently $-k = (0, k)_\sim$.

Task 2.10

Define integer multiplication and show that the multiplication is well-defined.

Task 2.11

Show that $1 = (1, 0)_\sim$ is the neutral element with respect to multiplication.

It is not hard to show that multiplication is commutative and associative. Moreover the distributive law holds in $\mathbb{Z}$.

Task 2.12

With ϕ as defined in Task 2.9, show that

$$\phi(m) \cdot \phi(n) = \phi(m \cdot n).$$

Last not least we will define a relation on $\mathbb{Z}$ as follows:

$$m \leq n \text{ if and only if } n + (-m) \in \mathbb{N}.$$

Task 2.13
Let $a, b, c, d \in \mathbb{N}$. Then $(a, b)_\sim \leq (c, d)_\sim$ if and only if there is a $k \in \mathbb{N}$ such that

$$(a + k, b) \sim (c, d).$$

The next four tasks show that "$\leq$" is a **total order** on $\mathbb{Z}$:

Task 2.14
Show that "$\leq$" is reflexive on $\mathbb{Z}$.

Task 2.15
Show that "$\leq$" is anti-symmetric on $\mathbb{Z}$.

Task 2.16
Show that "$\leq$" is transitive on $\mathbb{Z}$.

Task 2.17
$m \leq n$ or $n \leq m$ for all $m, n \in \mathbb{Z}$.

Task 2.18
If $m \leq n$, then $m + k \leq n + k$ for all $k \in \mathbb{Z}$.

Task 2.19
If $m \leq n$ and $0 \leq k$, then $m \cdot k \leq n \cdot k$.

Task 2.20
With ϕ as defined in Task 2.9 and $m, n \in \mathbb{N}$, show that $m \leq n$ if and only if $\phi(m) \leq \phi(n)$.

3 The Rational Numbers

Once again we define the next larger set as certain equivalence classes. On $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$, we define a relation $\cong$ as follows:

$$(a, b) \cong (c, d) \text{ if and only if } a \cdot d = b \cdot c.$$

We write equivalence classes in the familiar way

$$\frac{a}{b} = \{(c, d) \mid (c, d) \cong (a, b)\},$$

and denote the rational numbers by

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus \{0\} \right\}.$$

For integers n we write n instead of $\frac{n}{1}$.

We define addition and multiplication by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Additionally, we define an order on $\mathbb{Q}$ as follows:

$$0 \leq \frac{a}{b} \text{ if and only if } (0 \leq a \text{ and } 0 < b) \text{ or } (a \leq 0 \text{ and } b < 0).$$

For $p, q \in \mathbb{Q}$, we write $p \leq q$ if $0 \leq q - p$.[13] We also write $q \geq p$ when $p \leq q$.

With these arithmetic operations and the order above, the set of rational numbers becomes an **ordered field**:

Theorem 3.1. $(\mathbb{Q}, +, \cdot, \leq)$ has the following properties:

1. $(\mathbb{Q}, +)$ is an Abelian group with neutral element 0.
2. $(\mathbb{Q} \setminus \{0\}, \cdot)$ is an Abelian group with neutral element 1.
3. $(a + b) \cdot c = a \cdot c + b \cdot c$.
4. $(\mathbb{Q}, \leq)$ is a total order.
5. (a) $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in \mathbb{Q}$.
   (b) $a \leq b$ implies $a \cdot c \leq b \cdot c$ for all $a, b, c \in \mathbb{Q}$ with $0 \leq c$.

Let us write $a < b$ if $a \leq b$ and $a \neq b$. We will say that a is *positive*, if $a > 0$. Similarly, a is called *negative*, if $-a > 0$.

Task 3.1
Let $a, b \in \mathbb{Q}$, and assume $a > b$ and $b > 0$. Then $a^2 > b^2$.

[13] Subtraction is defined as $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.

Task 3.2
$\mathbb{Q}$ is *dense in itself*: For all $a, b \in \mathbb{Q}$ with $a < b$ there is a $c \in \mathbb{Q}$ with $a < c < b$.