A Primer on Equivalence Relations

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Let A be a set. A *relation* $\sim$ on A is a subset of the Cartesian product $A \times A$. Instead of writing $(a, b) \in \sim$, we will write $a \sim b$.

A relation $\sim$ on A is called an *equivalence relation* if it satisfies the following three properties:

(Reflexivity) For all $a \in A$, $a \sim a$.

(Symmetry) If $a \sim b$, then $b \sim a$.

(Transitivity) If $a \sim b$ and $b \sim c$, then $a \sim c$.

Given an element $a \in A$, its *equivalence class*, denoted by $a_\sim$, is defined as

$$a_\sim = \{b \in A \mid a \sim b\}.$$

$A_\sim$ will denote the set of all equivalence classes:

$$A_\sim = \{a_\sim \mid a \in A\}.$$

Since each equivalence class is a subset of A (and thus an element of its power set $\mathcal{P}(A)$), we obtain that $A_\sim \subseteq \mathcal{P}(A)$.

Given a set A, we say that $\mathcal{B} \subseteq \mathcal{P}(A)$ is a *partition* of A, if it satisfies the following three properties:

(P1) $B \neq \emptyset$ for all $B \in \mathcal{B}$.

(P2) $B \cap C = \emptyset$ or $B = C$ for all $B, C \in \mathcal{B}$.

(P3) $\displaystyle\bigcup_{B \in \mathcal{B}} B = A$.

Problem 1 If $\sim$ is an equivalence relation on a set A, then the set of its equivalence classes $A_\sim$ is a partition of A.

Given a partition $\mathcal{B} \subseteq \mathcal{P}(A)$, we can define a relation $\sim$ on A as follows:

$$a \sim b \text{ if there is a } B \in \mathcal{B} \text{ with } a \in B \textbf{ and } b \in B.$$

We will call this the *relation induced by the partition* $\mathcal{B}$.

Problem 2 Show that this relation is an equivalence relation.

Problem 3 Let $\sim$ be an equivalence relation. Let $\sim^*$ be the equivalence relation induced by the partition of equivalence classes of $\sim$. Then $\sim^* = \sim$.

Problem 4 Let $\mathcal{B}$ be a partition of A, and let $\sim$ be its induced equivalence relation. Then $\mathcal{B} = A_\sim$.

Problem 5 Consider the relation $\sim$ on $\mathbb{Z}$ defined as follows: $a \sim b$ if 5 divides evenly into $(a - b)$. Show that $\sim$ is an equivalence relation.

Problem 6 For this relation find the set of equivalence classes $\mathbb{Z}_\sim$. How many distinct elements does $\mathbb{Z}_\sim$ have?

Problem 7 For $x \in \mathbb{R}$, let

$$x + \mathbb{Q} = \{x + q \mid q \in \mathbb{Q}\}.$$

Show that

$$\mathcal{B} = \{x + \mathbb{Q} \mid x \in \mathbb{R}\}$$

is a partition of $\mathbb{R}$.

Problem 8 Determine the equivalence relation induced by this partition.

Problem 9 Consider the following equivalence relation $\sim$ on $\mathbb{R}$:

$$x \sim y \quad \Leftrightarrow \quad x^3 - 3x = y^3 - 3y.$$

1. Find the equivalence class $5_\sim$.
2. Find the equivalence class $0_\sim$.
3. Find all $x \in \mathbb{R}$ such that $x_\sim$ has exactly 2 elements. (This shows that $f(x) = x^3 - 3x$ is not injective.)
4. Define a new function $g : A_\sim \to \mathbb{R}$ by setting $g(x_\sim) = f(x)$. Show that g is well-defined and injective.