Math 4303 **Worksheet #1** **January 30, 2025**

Suppose that addition and multiplication have already been defined for the set of natural numbers $\mathbb{N}$.

1. We define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(p, q) \sim (p', q') \Leftrightarrow p + q' = p' + q.$$

   Show that $\sim$ defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

2. It then makes sense to define equivalence classes $(p, q)_\sim$:

$$(p, q)_\sim := \{(p', q') \in \mathbb{N} \times \mathbb{N} \mid (p', q') \sim (p, q)\}.$$

   The set of all these equivalence classes is denoted by $(\mathbb{N} \times \mathbb{N})_\sim$.

   Find all elements in the equivalence class $(2, 5)_\sim$. What do all these pairs of natural numbers have in common?

   Find all elements in the equivalence class $(4, 2)_\sim$. What do all these pairs of natural numbers have in common?

3. One can then identify the set of integers $\mathbb{Z}$ with this set $(\mathbb{N} \times \mathbb{N})_\sim$. Which equivalence class corresponds to the integer 0? What about the equivalence classes corresponding to the integers 1 and -3, respectively?

4. How can one define addition of two integers? More precisely, what should be the meaning of

$$(p, q)_\sim + (p', q')_\sim?$$

   Is your definition well-defined[1]?

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5. Show that addition as defined in 4. is commutative.

6. What is the neutral element in $(\mathbb{N} \times \mathbb{N})_\sim$ with respect to addition?

7. Given $(p, q)_\sim \in (\mathbb{N}\times\mathbb{N})_\sim$, what is the inverse element of $(p, q)_\sim$ with respect to addition?

8. How can one define multiplication of two integers? More precisely, what should be the meaning of

$$(p, q)_\sim \cdot (p', q')_\sim?$$

   Is your definition well-defined?

9. Verify that $(2, 5)_\sim \cdot (1, 2)_\sim = (5, 2)_\sim$.

[1] You have to check that your definition does not depend on the representatives chosen from the two equivalence classes.