Here is our system of four equations:

$$h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \tag{1}$$

$$h_0 h_2 + h_1 h_3 = 0 \tag{2}$$

$$h_0 - h_1 + h_2 - h_3 = 0 \tag{3}$$

$$h_1 - 2h_2 + 3h_3 = 0 \tag{4}$$

We first solve (3) and (4) for h_2 and h_3.

$$h_2 = -h_0 + h_1 + h_3 \tag{5}$$

Substituting (5) in the last equation yields

$$h_1 - 2(-h_0 + h_1 + h_3) + 3h_3 = 0 \tag{6}$$

and thus

$$\boxed{h_3 = h_1 - 2h_0} \tag{7}$$

Going back to (5) we obtain:

$$\boxed{h_2 = 2h_1 - 3h_0} \tag{8}$$

Next we substitute (7) and (8) in (2) and solve for h_1:

$$h_0(2h_1 - 3h_0) + h_1(h_1 - 2h_0) = 0 \tag{9}$$

This yields

$$h_1^2 = 3h_0^2 \tag{10}$$

Discarding one of the two solutions we obtain

$$\boxed{h_1 = \sqrt{3}h_0} \tag{11}$$

Finally we substitute (7), (8) in (1):

$$h_0^2 + h_1^2 + (2h_1 - 3h_0)^2 + (h_1 - 2h_0)^2 = 1 \tag{12}$$

and obtain

$$14h_0^2 + 6h_1^2 - 16h_0 h_1 = 1 \tag{13}$$

Using (11) we get:

$$14h_0^2 + 18h_0^2 - 16\sqrt{3}h_0^2 = 1 \tag{14}$$

Solving for h_0 (again discarding one of the two solutions) we obtain:

$$h_0 = \frac{1}{4\sqrt{2-\sqrt{3}}} \tag{15}$$

Backsolving we can compute all 4 coefficients:

$$
\begin{aligned}
h_0 &= \frac{1}{4\sqrt{2-\sqrt{3}}} \\
h_1 &= \frac{\sqrt{3}}{4\sqrt{2-\sqrt{3}}} \\
h_2 &= \frac{2\sqrt{3}-3}{4\sqrt{2-\sqrt{3}}} \\
h_3 &= \frac{\sqrt{3}-2}{4\sqrt{2-\sqrt{3}}}
\end{aligned}
$$

Finally note that

$$
\begin{aligned}
& h_0 + h_1 + h_2 + h_3 \\
= \; & \frac{1}{4\sqrt{2-\sqrt{3}}}(1+\sqrt{3}+2\sqrt{3}-3+\sqrt{3}-2) \\
= \; & \frac{1}{4\sqrt{2-\sqrt{3}}}(-4+4\sqrt{3}) = \frac{\sqrt{3}-1}{\sqrt{2-\sqrt{3}}} \\
= \; & \frac{(\sqrt{3}-1)(\sqrt{2+\sqrt{3}})}{(\sqrt{2-\sqrt{3}})(\sqrt{2+\sqrt{3}})} \\
= \; & \sqrt{4-2\sqrt{3}}\sqrt{2+\sqrt{3}} \\
= \; & \sqrt{2}\sqrt{(2-\sqrt{3})(2+\sqrt{3})} = \sqrt{2}
\end{aligned}
$$