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$(\mathbb{R}, +, \cdot)$ form a field

In fact $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field

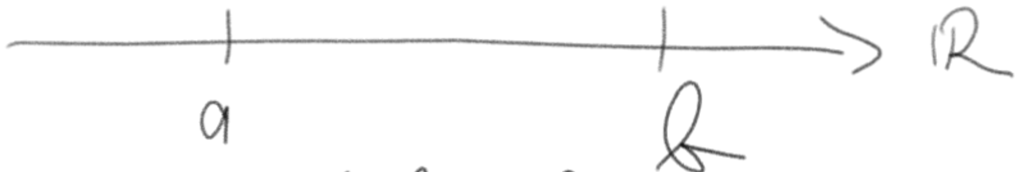
i.e. 1) $\leq$ is a total order on $\mathbb{R}$

a) $\leq$ is anti-symmetric

b) $\leq$ is transitive

c) $\leq$ is reflexive

d) $\forall a, b \in \mathbb{R}$: $a \leq b$ or $b \leq a$



e) $\forall a, b, c \in \mathbb{R}$: $a \leq b \Rightarrow a + c \leq b + c$

f) $\forall a, b \in \mathbb{R}$, $c \geq 0$; $a \leq b \Rightarrow ac \leq bc$

$\rightarrow$ $\mathbb{R}$ is complete

Completeness Axiom : If $A \subset \mathbb{R}$, $A \neq \emptyset$, A is bounded from above, then A has a supremum ($\sup A$)

Def : $\mathbb{R} \ni S$ is the $\sup A$

if 1) S is an upper bound of A

2) If y is an upper bound of A, then $y \geq S$.

(If $y < S$, then y is not an upper bound of A)

Examples 1) $A = (0, 1) = \{ x \in \mathbb{R} \mid 0 < x < 1 \}$

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$\sup A = 1 \ (\notin A)$

2) $A = [0,2]$

$\max A = 2 \ (= \sup A)$

3) $A = \{2 - \frac{1}{n} \mid n \in \mathbb{N}\}$

$\sup A = 2 \ (\notin A)$

4) $\mathbb{R}$ is not bounded so $\mathbb{R}$ has no sup!

Exercise a) a set can have at most one supremum.

b) If a set has a maximum, the sup is equal to the maximum.

Consequences of Completeness Axiom

Nested Interval Property

Suppose (A_n) is a sequence of closed bounded intervals, i.e. $A_n = [a_n, b_n]$ for some $a_n, b_n \in \mathbb{R}$ and assume

$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$

then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$

Version for HS: If additionally $b_n - a_n \to 0$ then $\bigcap_{n=1}^{\infty} A_n$ consists of exactly one point.

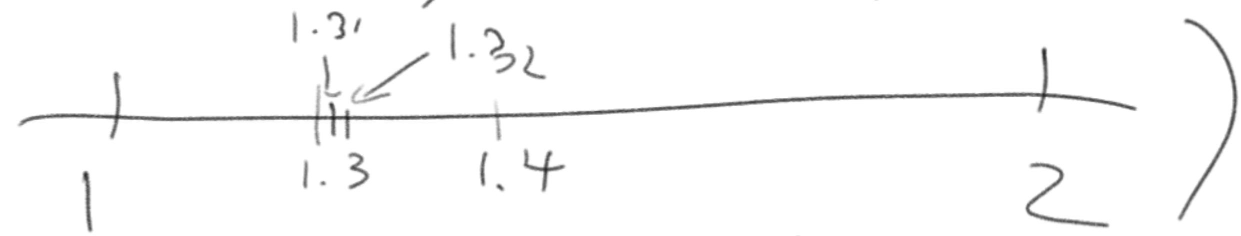
Correspondence b/w the set of real numbers and the $\mathbb{R}$ number line

$x = 1.315704\ldots$

$\hookrightarrow A_1 = [1,2]$

$A_2 = [1.3, 1.4]$

$A_3 = [1.31, 1.32]$



Remarks ① NIP not true within Q:

$A_1 = [1,2] \cap Q$

$A_2 = [1.4, 1.5] \cap Q$

$A_3 = [1.41, 1.42] \cap Q$

$\bigcap_{n=1} A_n = \emptyset$ in Q

② NIP does not if A_n's are not closed

Ex: $A_n = (0, \frac{1}{n})$

A_n bdd ✓, A_n nested ✓

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

③ NIP does not work if A_n's are not bounded

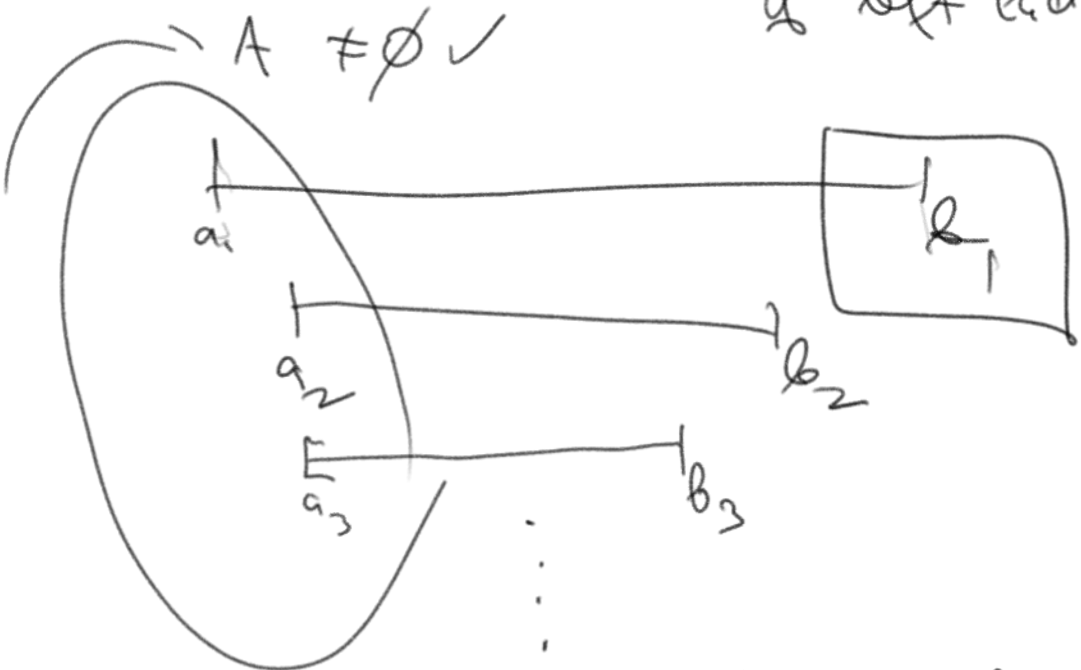
$A_n = [n, \infty)$ closed, unbounded

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

Pf of NIP given $A_n = [a_n, b_n]$

we consider $A = \{a_n \mid n \in \mathbb{N}\}$ the set we consider $A = \{a_n \mid n \in \mathbb{N}\}$ the set of left endpoints

$\Rightarrow A \neq \emptyset$ ✓



A is bounded above by b_1

thus A has a supremum, call it s.

Claim: $s \in \bigcap_{n=1}^{\infty} A_n$

1) $s \geq a_n \ \forall n \in \mathbb{N}$ ✓ s is an upper bound of A

2) $s \leq b_n \ \forall n \in \mathbb{N}$

suppose false: then there will be an $n \in \mathbb{N}$ such that $s > b_n$.

this can't be true:

b_n is an upper bound for A and $b_n < s$

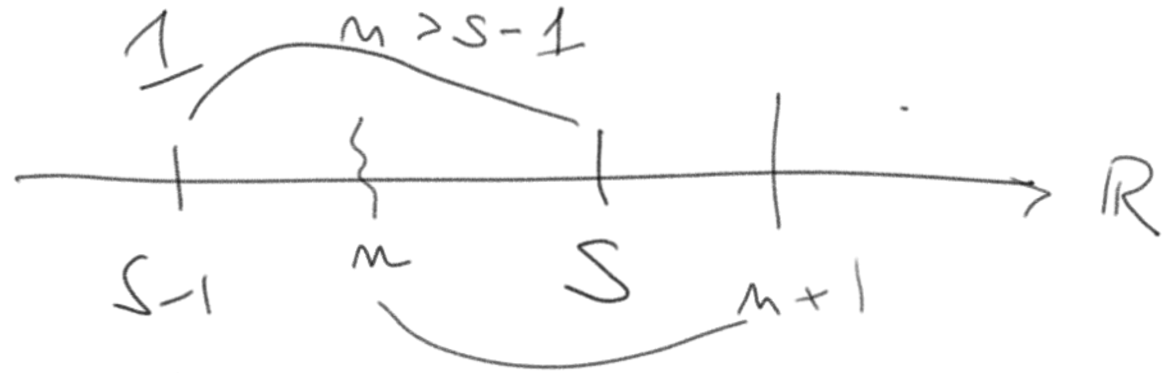
XX

$s_n < S$ XX

Archimedean Principle

$\mathbb{N}$ is not Bounded,
i.e. for every real number r
there is a natural number n
with $n > r$.

Pf. Suppose to the contrary that $\mathbb{N}$ is Bounded,
there is an $r \in \mathbb{R}$ s. that
$r > n$ for all $n \in \mathbb{N}$
By the completeness axiom, $\mathbb{N}$ has
a supremum, call is S $(\in \mathbb{R})$
By the lemma below,
$S-1$ will not be an upper bound
for $\mathbb{N}$
thus there is an $n \in \mathbb{N}$ s. that
$n > S-1$

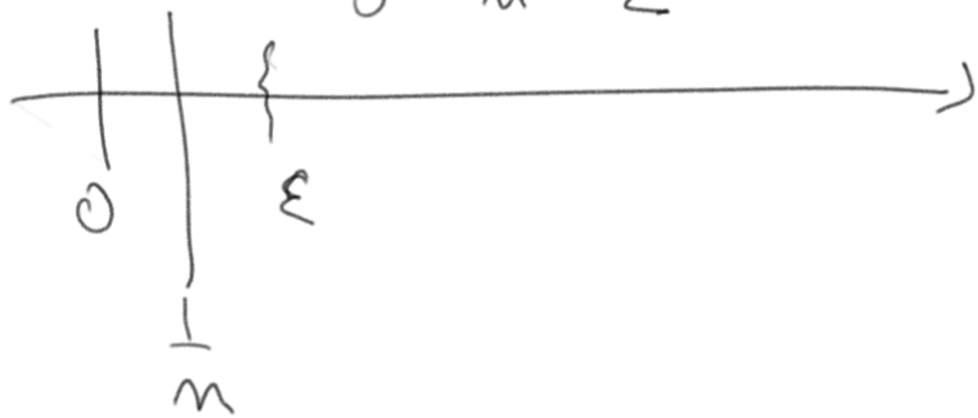


then $n+1 > S$. ($n+1 \in \mathbb{N}$
Note)
a contradiction

Lemma: let S be the supremum of A,
let $\varepsilon > 0$. then there is an
element a in A with $a > S - \varepsilon$

element/a in A with $a > s - \varepsilon$.

Corollary: For all real $\varepsilon > 0$, $(a \geq s)$
there is a natural number n
s. that $0 < \frac{1}{n} < \varepsilon$



Pf (using Archie)
let n be a natural number $> \frac{1}{\varepsilon}$

$$n > \frac{1}{\varepsilon} \iff \frac{1}{n} < \varepsilon .$$

$\mathbb{Q}$ is dense in $\mathbb{R}$

given $a, b \in \mathbb{R}$ with $a < b$
then there is a $q \in \mathbb{Q}$
with $a < q < b$

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\}$$

Pf: $b - a > 0$. So by the Corollary to Archie there is an $n \in \mathbb{N}$ s.that

$$\frac{1}{n} < b - a$$

want to find $k \in \mathbb{N}$ such that

$$\longrightarrow \quad na < k < nb$$

$$(a < \frac{k}{n} < b)$$

Archie again:

Saved by the bell...