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Tuesday, September 2, 2025 14:45

Recall AP: $\mathbb{N}$ is unbounded:

$$\forall r\in\mathbb{R}\ \exists n\in\mathbb{N}: n > r$$

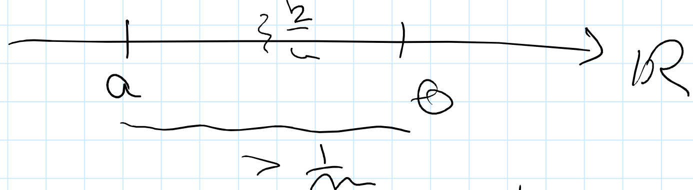
Corollary $\forall r>0\ \exists n\in\mathbb{N}: \quad 0<\frac{1}{n}<r$

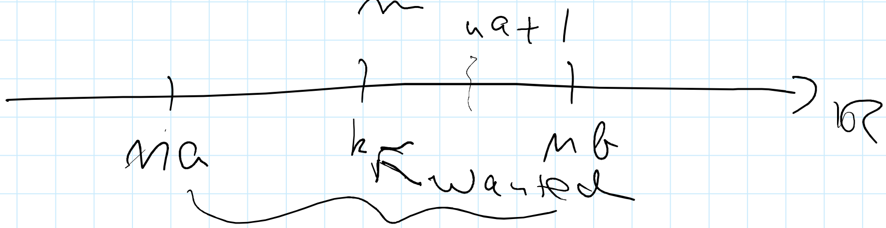
Theorem $\mathbb{Q}$ is dense in $\mathbb{R}$:

$$\forall a,b\in\mathbb{R},\ a<b\ \exists q\in\mathbb{Q}: a<q<b$$

pf Let $a<b$ be given.

There is an $n\in\mathbb{N}$ s. that $b-a>\frac{1}{n}$





let $K=\{k\in\mathbb{Z} \mid k>na\}$

$K\neq\emptyset$, K is bounded from below

thus K has an infimum, call it s.

Consequently $s+1$ is not a lower bound of K

so there is a $m\in K$ with $m<s+1$

$m-1<s$ thus $m-1\notin K$

$$m-1\leq na$$

$m - 1 \le na$

thus $m \le na + 1 < nb$

$m \in K \Rightarrow m > na$

Theorem $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, i.e.

given $a < b$ there is a $q \in \mathbb{R} \setminus \mathbb{Q}$

$a < q < b$

Pf Given $a < b$, $a, b \in \mathbb{R}$

$$a\sqrt{2} < b\sqrt{2}$$

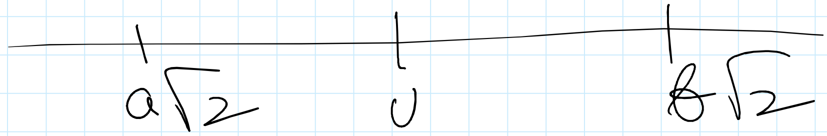
By previous result, there is $r \in \mathbb{Q}$ s. that $a\sqrt{2} < r < b\sqrt{2}$

so $a < \frac{r}{\sqrt{2}} < b$

since $r \in \mathbb{Q}$ and $\sqrt{2} \notin \mathbb{Q}$, $\underbrace{\frac{r}{\sqrt{2}}}_{= q} \notin \mathbb{Q}$ and done.

What's wrong with this proof?

only problematic case is $r = 0$ $\cancel{q}$



Consider $0 < b\sqrt{2}$

We know there is a $r \in \mathbb{Q}$

$a\sqrt{2} < 0 < r < b\sqrt{2}$

so $a < 0 < \frac{r}{\sqrt{2}} < b$ and done.

→ go to 2.2 in the book

Definition A (real-valued) sequence is a function $\phi : \mathbb{N} \to \mathbb{R}$

[illegible] sequence

is a function $\phi : \mathbb{N} \to \mathbb{R}$

Exa $\phi(n) = \frac{1}{n^2}$

$\phi(1) = 1 \quad \phi(2) = \frac{1}{4} \quad \phi(3) = \frac{1}{9} \quad \ldots$

usually people write

$(x_n)_{n\in\mathbb{N}} = \left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$

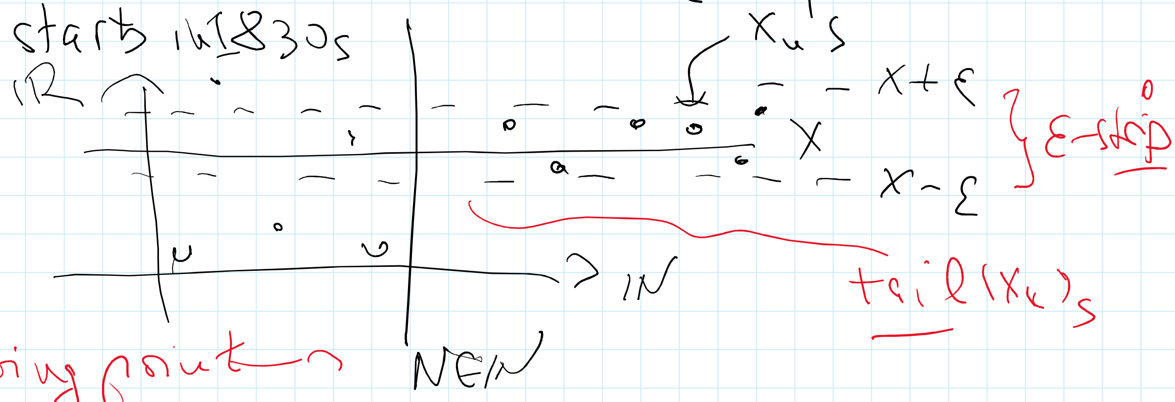
$x_1 = 1, \ x_2 = \frac{1}{4}, \ x_3 = \frac{1}{9}, \ \ldots$

$(x_n)_{n\in\mathbb{N}}$ write $x_n = \frac{1}{n^2}$

Definition: given a sequence $(x_n)_{n\in\mathbb{N}}$ and $x \in \mathbb{R}$, we say (x_n) converges to x if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N$

$$|x_n - x| < \varepsilon$$

Analysis starts in 1830s



Exa $(x_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

Claim $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ converges to 0

pf: let $\varepsilon > 0$. let $N \in \mathbb{N}$ be such that item $\frac{1}{N} < \varepsilon$ (by Cor to AP)

motivate

$$\frac{1}{N} < \varepsilon \quad \text{(By Cor to AP)}$$

Now consider $n \geq N$.

then $\frac{1}{n} \leq \frac{1}{N}$

But $|x_n - x| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

check

and done.

Exa 2 $\quad x_n = \frac{n^2+1}{n^2} \qquad x = 1$

Aside

What do we have to show in the end?

$$|x_n - x| = \left|\frac{n^2+1}{n^2} - 1\right| \overset{!}{<} \varepsilon$$

$$\frac{1}{n^2} = \left|1 + \frac{1}{n^2} - 1\right|$$

need $\frac{1}{n^2} < \varepsilon \iff n > \frac{1}{\sqrt{\varepsilon}}$

Pf Let $\varepsilon > 0$ be given. Pick $N \in \mathbb{N}$ satisfying $N > \frac{1}{\sqrt{\varepsilon}}$. Consider $n \geq N$ then

$$|x_n - x| = \left|\left(1 + \frac{1}{n^2}\right) - 1\right| = \frac{1}{n^2} < \frac{1}{N^2} < \varepsilon$$

and done.

Definition We say that (x_n) converges if $\exists x \in \mathbb{R}$ s. that (x_n) converges to x.

Definition If (x_n) does not converge, we say (x_n) is divergent.

$(x_n) \to x$! $\quad \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : |x_n - x| < \varepsilon$

$(x_n) \to x$: $\forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\geq N : |x_n - x| < \varepsilon$

(x_n) converges: $\exists x\in\mathbb{R}\ \forall\varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\geq N$ $|x_n - x|<\varepsilon$

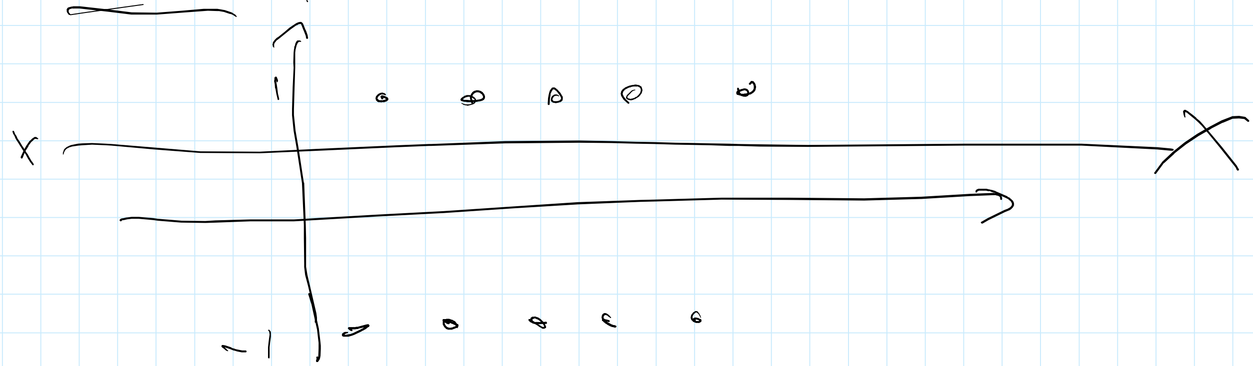
(x_n) diverges: $\forall x\in\mathbb{R}\ \exists\varepsilon>0\ \forall N\in\mathbb{N}$
$\exists n\geq N : |x_n - x| \geq \varepsilon$

Exa $(x_n) = (-1)^n$

$x_1 = -1,\ x_2 = 1,\ x_3 = -1,\ x_4 = 1, \ldots$

Pf Let $x\in\mathbb{R}$

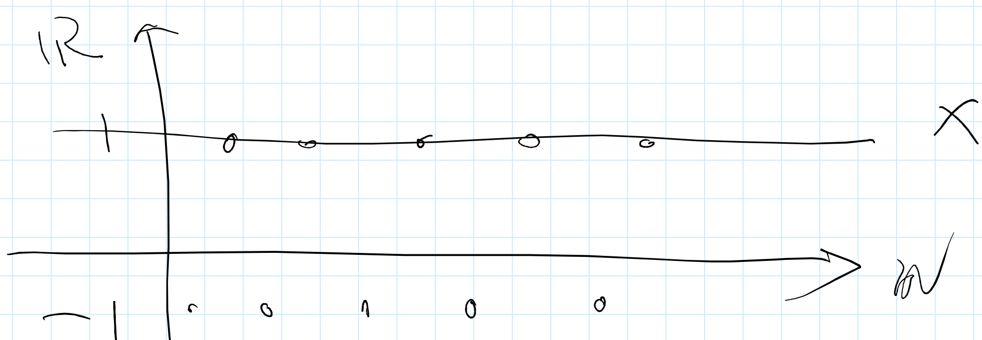
Case 1 $x\neq 1$



Let $\varepsilon = |1-x| > 0$. Now let $N\in\mathbb{N}$.
Pick $n\geq N$, n even

then $|x_n - x| = |1-x| \geq \varepsilon$ ✓

Case 2 $x=1$



Let $\varepsilon = 1$ Let $N\in\mathbb{N}$ be given let $\varepsilon = 1$ let $N \in \mathbb{N}$ be given

Choose $n \geq N$ odd

then

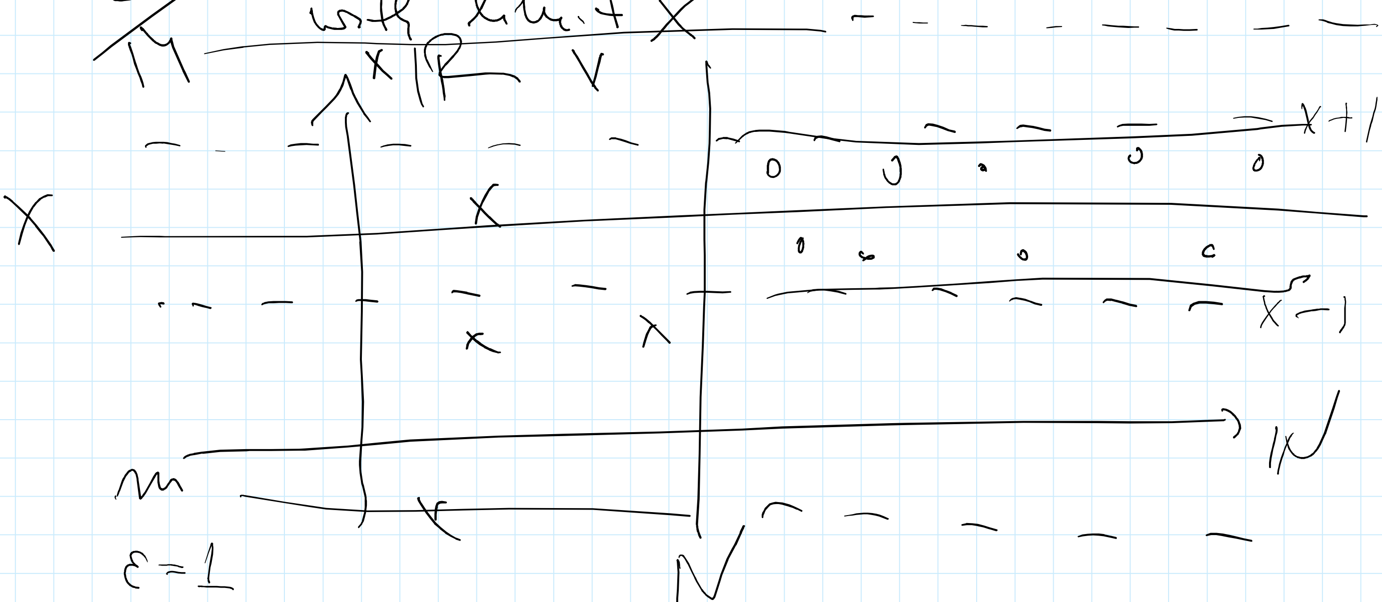
$|x_n - x| = |(-1) - 1| = 2 \geq 1 = \varepsilon$

Theorem Every convergent sequence is bounded

Def (x_n) is bounded if its range is bounded, i.e. $\exists M \in \mathbb{R}$

$|x_n| \leq M \quad \forall n \in \mathbb{N}$

~~Pf~~ let (x_n) be a convergent sequence with limit x



$\varepsilon = 1$

thus for all $n \geq N$

$x - 1 < x_n < x + 1$

let $M = \max\{x_1, x_2, \ldots, x_{N-1}\}$

$m = \min\{x_1, x_2, \ldots, x_{N-1}\}$

thus for all $n \in \mathbb{N}$

$N-1$)

thus for all $n \in \mathbb{N}$

$$x_n \leq \max\{x+1, M\}$$

and $x_n \geq \min\{x-1, m\}$

so (x_n) is bounded.