Last result last time:

$\longrightarrow$ converging sequences are bounded

Exa $x_n = n \ \forall n \in \mathbb{N}$

(x_n) is unbounded and thus divergent.

Algebra of Limits

Theorem $x_n \to x$, $y_n \to y$

then $(x_n + y_n) \to x + y$

Pf [Aside:

$$|(x_n + y_n) - (x+y)| = |(x_n - x) + (y_n - y)|$$

$$\leq |x_n - x| + |y_n - y| < \varepsilon$$

Let $\varepsilon > 0$ be given

Since (x_n) converges to x,
can find $N_1 \in \mathbb{N}$ s.t. for all $n \geq N_1$

$$|x_n - x| < \varepsilon/2$$

Similarly, can find $N_2 \in \mathbb{N}$ s.t.
for all $n \geq N_2$

$$|y_n - y| < \varepsilon/2$$

Let $N = \max\{N_1, N_2\}$ and consider $n \geq N$ (then $n \geq N_1$ and $n \geq N_2$)

Then $|(x_n + y_n) - (x+y)|$

$$\leq |x_n - x| + |y_n - y|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

2nd Theorem if $x_n \to x$ and $y_n \to y$,
then $x_n \cdot y_n \to xy$

Theorem: if $x_n \to x$ and $y_n \to y$,
then $x_n y_n \to xy$.

Aside:
$$|x_n y_n - xy|$$
$$= |x_n y_n - x_n y + x_n y - xy|$$
$$= |(x_n y_n - x_n y) + (x_n y - xy)|$$
$$\leq |x_n y_n - x_n y| + |x_n y - xy|$$
$$= |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x|$$
bound for (x_n)
$$\leq M \cdot |y_n - y| + |y|\,|x_n - x|$$
$$< M \cdot \varepsilon' + |y| \cdot \varepsilon'' \leq \varepsilon$$

Pf: Let $\varepsilon > 0$ be given.
Let $M > 0$ be chosen such that $|x_n| \leq M$ for all n.
Let $\varepsilon' = \dfrac{\varepsilon}{2M}$ and $\varepsilon'' = \cancel{\dfrac{\varepsilon}{2|y|}} = \dfrac{\varepsilon}{2\max\{1, |y|\}}$

We can find N_1 s.t. that
$$|y_n - y| < \varepsilon' \text{ for all } n \geq N_1$$
and can find N_2 such that
$$|x_n - x| < \varepsilon'' \text{ for all } n \geq N_2$$
Let $N = \max\{N_1, N_2\}$ and consider $n \geq N$.

Then $|x_n y_n - xy|$
$$\leq |x_n| \cdot |y_n - y| + |y|\,|x_n - x|$$
$$< M \cdot \varepsilon' + |y| \cdot \varepsilon''$$
$$\leq M \cdot \frac{\varepsilon}{2M} + |y| \frac{\varepsilon}{2|y|} = \varepsilon$$

$\leq M \cdot \frac{\varepsilon}{2M} + |y| \frac{\varepsilon}{2|y|} = \varepsilon$

what's wrong with the proof ?

→ can't divide by 0

Theorem 3 $y_n \to y$, $y \neq 0$, $y_n \neq 0$ for all $n \in \mathbb{N}$

then $\frac{1}{y_n} \to \frac{1}{y}$

Aside $\left| \frac{1}{y_n} - \frac{1}{y} \right| = \frac{|y - y_n|}{|y_n| \cdot |y|}$ need $< \varepsilon$

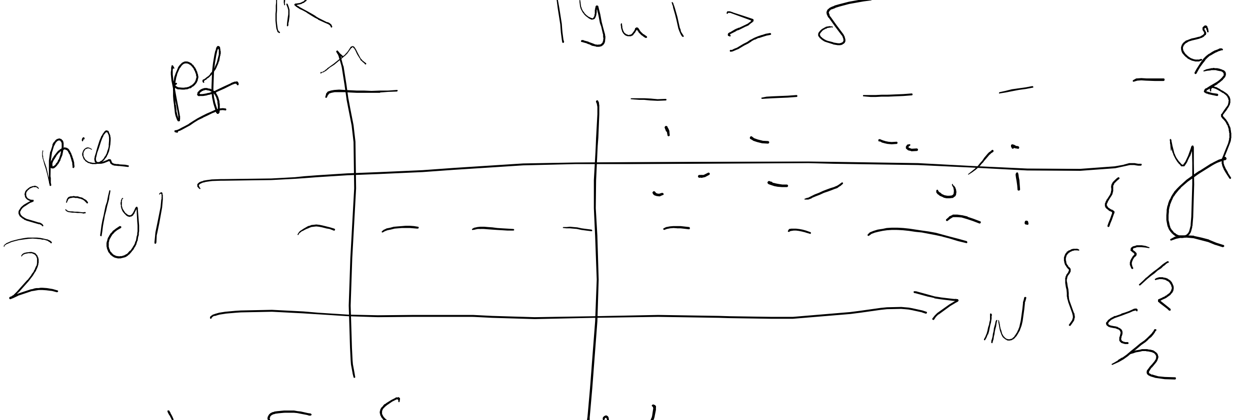
worrisome

Lemma $y_n \to y$, $y \neq 0$, $y_n \neq 0$ $\forall n$

Then there is a $\delta > 0$ and $N \in \mathbb{N}$ such that for $n \geq N$

$|y_n| \geq \delta$

Pf



let $\delta = \varepsilon/2$

pf Set $\varepsilon = \frac{|y|}{2}$. Let $\delta = \varepsilon > 0$

then there is an $N \in \mathbb{N}$

s.t. for $n \geq N$: $|y_n - y| < \varepsilon = |y|$

then there is an $N \in \mathbb{N}$

s.t. for $n \geq N$: $|y_n - y| < \varepsilon = \frac{|y|}{2}$

$\Rightarrow |y_n| \geq \frac{|y|}{2} = \delta$

triangle inequality

pf of Thm 3 Let $\varepsilon > 0$ be given.

Pick $\varepsilon' = \delta |y| \varepsilon$ ($y \neq 0$!)

where δ is picked according to the previous lemma with accompanying N:

$|y_n| \geq \delta$ for all $n > N$

For ε' we can find N_1 such that

for $n \geq N_1$: $|y_n - y| < \varepsilon'$

thus for $n \geq \max\{N, N_1\}$

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \frac{|y - y_n|}{|y_n||y|} < \frac{\varepsilon'}{\delta |y|} = \varepsilon.$$

If $|y_n| \geq \delta \Rightarrow \frac{1}{|y_n|} \leq \frac{1}{\delta}$

Theorem 3A $x_n \to x$, $y_n \to y$

$y \neq 0$, $y_n \neq 0$ $\forall n$

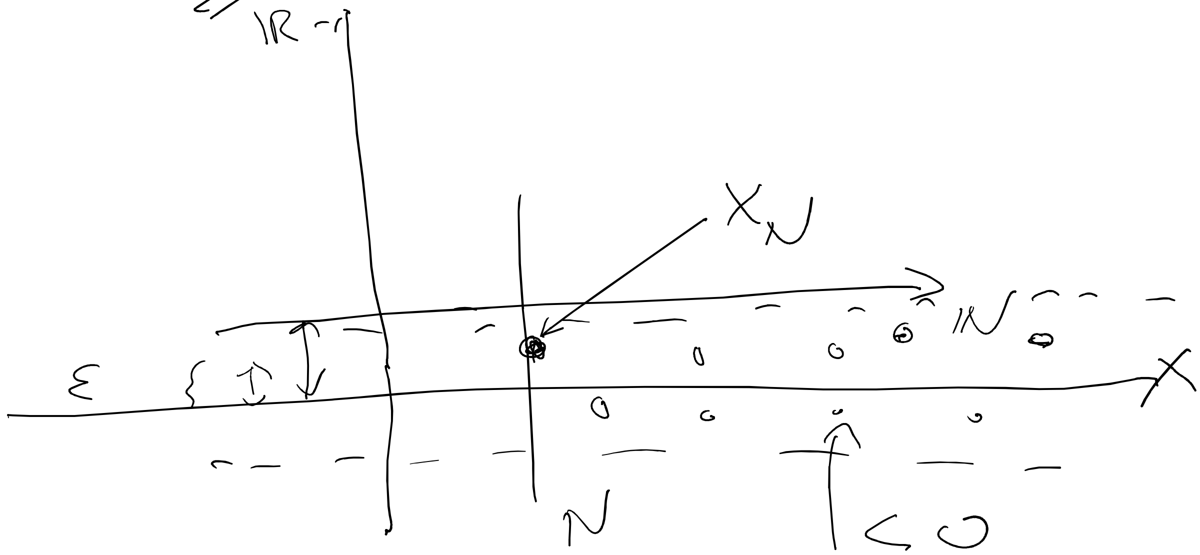
then $\frac{x_n}{y_n} \longrightarrow \frac{x}{y}$

Pf: Combine theorems 2 and 3

Theorem If $x_n \to x$ and $x_n \geq 0 \; \forall n$ then $x \geq 0$

(not true: $x_n > 0 \; \forall n \Rightarrow x > 0$)

Pf Suppose to the contrary that $x < 0$



let $\varepsilon = \frac{|x|}{2}$ then we can find $N \in \mathbb{N}$ s.t. that for all $n \geq N$

$$|x_n - x| < \varepsilon = \frac{|x|}{2}$$

in particular this work for $n = N$

$$|x_N - x| < \frac{|x|}{2}$$

$$\Rightarrow \quad x_N < 0$$

Theorem 4A

If $x_n \leq y_n \; \forall n \in \mathbb{N}$

$x_n \to x$, $y_n \to y$

then $x \leq y$

then $x \leq y$

~~Pf~~ ~~Set~~ $c_n = y_n - x_n \geq 0$

$0 \leq c \leftarrow c_n \to y - x \geq 0$

Application: let $P: \mathbb{R} \to \mathbb{R}$ be a polynomial, and let $x_n \to x$

Then $P(x_n) \longrightarrow P(x)$

"P is continuous at x"

Aside

what's a polynomial?

$\to$ A polynomial is a function

$\to$ A polynomial is a linear combination of powers of x

i.e. $P(x) = \sum_{n=0}^{k} a_n x^n$, $a_n \in \mathbb{R}$, $k \in \mathbb{N}$

direct consequence of thms 1 & 2

noting

If $x_n = x$ $\forall n \in \mathbb{N}$

then $x_n \longrightarrow x$

Monotone Convergence Theorem

$$\boxed{\text{If } (x_n) \text{ is increasing and bounded, then } (x_n) \text{ converges}}$$

Def (x_n) is increasing if

Def (x_n) is increasing if

$$m \le n \implies x_m \le x_n$$

Claim: $\lim_{n \to \infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$