MCT Monotone Convergence theorem

If (x_n) is increasing and bounded, then it converges.

bounded is necessary,

Ex Fibonacci

$x_1 = 1, \ x_2 = 1, \ x_3 = 2, \ x_4 = 3 \dots$

increasing is necessary

Ex

$x_1 = -1, \ x_2 = 1, \ x_3 = -1, \dots$

kicker: $\lim\limits_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$

pf let $A = \{x_n \mid n \in \mathbb{N}\}$. $A \neq \emptyset$, A is bounded from above (b/c (x_n) is bounded)

By CA, there is an $s \in \mathbb{R}$, $s = \sup A$

Claim (x_n) converges to s.

Indeed, let $\varepsilon > 0$, $s - \varepsilon$ is not an upper bound for A

i.e. $\exists N \in \mathbb{N}$ such that $x_N > s - \varepsilon$

Now consider $n \geq N$

then $x_n \geq x_N > s - \varepsilon$

Clearly we also have that $x_n \leq s$

so $|x_n - s| < \varepsilon$.

Ex 1) $x_n = \left(1 + \frac{1}{n}\right)^n$

fact: 1) (x_n) is increasing

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2) (x_n) is bounded by 3
MCT says
that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ exists
Euler's name for this limit was e

2) Consider the following sequence:

$$\begin{cases} a_1 = \sqrt{2} \\ a_{n+1} = \sqrt{2+\sqrt{a_n}} \quad \forall n \geq 1 \end{cases}$$

Claim 1 (a_n) is increasing

$n=1$ $a_2 = \sqrt{2+\sqrt{a_1}} = \sqrt{2+\sqrt{\sqrt{2}}} \geq \sqrt{2} = a_1$

$n \to n+1$ Suppose $a_n \geq a_{n-1}$
Want to show $a_{n+1} \geq a_n$:

$$a_{n+1} = \sqrt{2+\sqrt{a_n}} \geq \underbrace{\sqrt{2+\sqrt{a_{n-1}}}}_{= a_n}$$

Claim 2 (a_n) is bounded by 2

$n=1$ $a_1 = \sqrt{2} \leq 2$

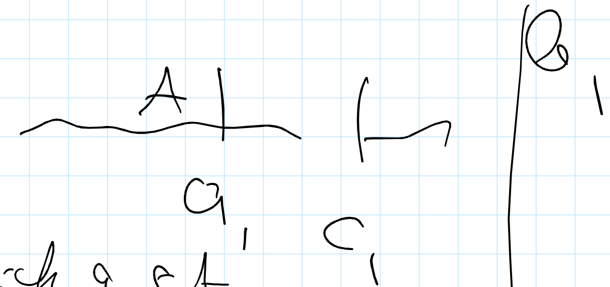
$n \to n+1$ suppose $a_n \leq 2$
then $a_{n+1} = \sqrt{2+\sqrt{a_n}}$
$\leq \sqrt{2+\sqrt{2}} \leq \sqrt{4} = 2$

therefore (a_n) converges.

We have shown: CA $\Rightarrow$ MCT

theorem: MCT $\Rightarrow$ CA

Outline of pf: Let $A \neq \emptyset$, bounded.



Can pick $a_1 \in A$

b_1 an upper bound for A

If A has a sup, it will be b/w a_1 and b_1

Strategy: find a_2 and b_2, closer to each other, and still the "sup" in between!

Let $c_1 = \frac{a_1 + b_1}{2}$

Case1) c_1 is an upper bound for A

Then set $a_2 = a_1$ & $b_2 = c_1$

Still "sup" is b/w a_2 and b_2

and $b_2 - a_2 \leq \frac{b_1 - a_1}{2}$

Case2 c_1 is not an upper bound for A

Then we can find $a_2 \in A$, $a_2 \geq c_1$

we set $b_2 = b_1$

Again $b_2 - a_2 \leq \frac{b_1 - a_1}{2}$

and "sup" lies b/w a_2 and b_2

Continuing in this fashion, we produce

(a_n) increasing, bounded from above (b_1)

(b_n) decreasing, bounded from below (a_1)

(b_n) decreasing, bounded from below (by a_1)

By MCT both sequences converge

Say (a_n) cv. to a

(b_n) cv. to b

Moreover $a = b$ b/c $|b_n - a_n| \longrightarrow 0$

left to show: $a = \sup A$.

Subsequences

sequence $\phi : \mathbb{N} \to \mathbb{R}$

$\psi : \mathbb{N} \to \mathbb{N}$ strictly increasing (if $m < n$ then $\psi(m) < \psi(n)$)

then $\phi \circ \psi : \mathbb{N} \to \mathbb{R}$

is called a subsequence of ϕ

Example $\phi(n) = \frac{1}{n^2}$

$\psi(k) = 2k$

$\phi : \quad 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}$

$\quad\ \ a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

$a_2 = a_{k_1}$, $a_4 = a_{k_2}$

$\psi(1) = 2$

$a_{k_1} = \phi \circ \psi(1) = \phi(2)$

$a_{k_2} = \phi \circ \psi(2) = \phi(4)$

$a_{k_1} = \frac{1}{4} \quad a_{k_2} = \frac{1}{16} \quad a_{k_3} = \frac{1}{36}, \ \ldots$

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"throw away, keep the order, keep infinitely many items"

Exc $x_n = (-1)^n$ is ugly b/c it is divergent

$x_{2n-1} = (-1)^{2n-1} = -1$ is "beautiful" since it converges.

Q Does every sequence have a converging subsequence?

NO $x_n = n$. (x_n) is unbounded, thus div.

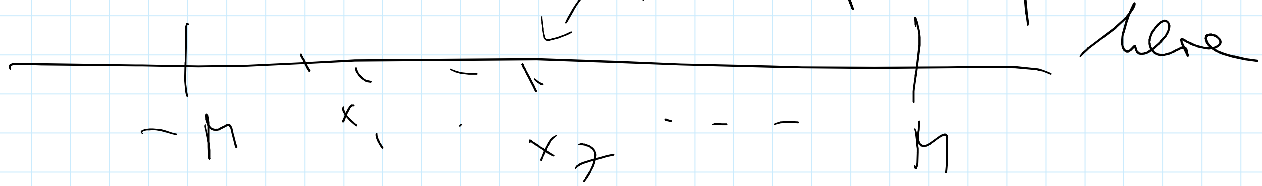
Every subsequence $x_{n_k} = n_k$ will also be unbounded, so divergent

Theorem [Every bounded sequence has a converging subsequence.

((Bolzano-Weierstrass-Theorem
(BW Theorem)))

pf Let (x_n) be a bounded sequence.

limit of subsequence here



Since (x_n) is bounded, there is an $M \in \mathbb{R}$ such that

$$-M \leq x_n \leq M \quad \forall n$$

Case 1 $\{n \mid x_n \in [0, M]\}$ is infinite

Set $a_1 = 0$, $b_1 = M$ and pick $x_{n_1} \in [0, M]$

Case 2 $\{n \mid x_n \in [0, M]\}$ is finite

Then $\{n \mid x_n \in [-M, 0]\}$ is infinite

Set $a_1 = -M$, $b_1 = 0$ and pick $x_{n_1} \in [-M, 0]$