Def $(x_n)_{n=1}^{\infty}$ a sequence.
If we pick $n_1 < n_2 < n_3 < \dots < n_k < \dots$
we call $(x_{n_k})_{k=1}^{\infty}$ a subsequence of (x_n).

BW thm Every bounded sequence has a converging subsequence.

Theorem If (x_n) converges, all of its subsequence converge as well

pf is based on the fact that
if $n_1 < n_2 < n_3 < n_4 \dots$
then $n_k \geq k \quad \forall k \in \mathbb{N}$

pf by induction
$k=1$ $\quad n_1 \geq 1$ ✓
$k \to k+1$ $\quad$ if we know $n_k \geq k$
then $n_{k+1} > n_k \geq k$
$\Rightarrow n_{k+1} \geq k+1$

N.B. All these subsequences will converge to the same limit

Application $x_n = (-1)^n$ does not converge!
consider the two subsequences
$(x_{2n}) = (1)$
$(x_{2n-1}) = (-1)$
they converge, albeit to

$(x_{2n-1}) = (-1)$ (converge to different limits

Pf (BW) (x_n) is bounded

<u>Step 1</u> so $\exists M > 0$ s.t. $x_n \in [-M, M]$ for all $n \in \mathbb{N}$

Consider the midpt 0.

<u>Case 1</u> $[-M, 0]$ contains infinitely many of the sequence elements. Pick such a seq. element, call it x_{n_1}

<u>Case 2</u> $[-M, 0]$ contains only finitely many sequence elements then $[0, M]$ contains infinitely many sequence elements, so we can pick $x_1 \in [0, M]$

Pigeon hole principle

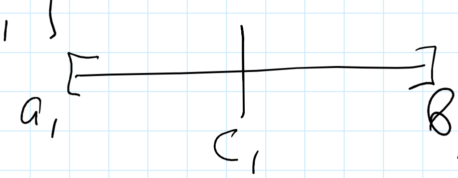
This will be a <u>condensation argument</u>

We call the new interval $[a_1, b_1]$

($[a_1, b_1] = [-M, 0]$ or $[0, M]$

At this point we have pick $x_{n_1} \in [a_1, b_1]$

<u>Step 2</u> let $c_1 = \frac{a_1 + b_1}{2}$



Case 1 $[a_1, c_1]$ contains infinitely many sequence elements

Since there are infinitely many choices we can $x_{n_2} \in [a_1, c_1]$ with $n_2 > n_1$. We call $[a_1, c_1] = [a_2, b_2]$

Case 2 $[c_1, b_1]$ contains infinitely many sequence elements, so we can pick $x_{n_2} \in [c_1, b_1] = [a_2, b_2]$ with $n_2 > n_1$

Note that

$$x_{n_2} \in [a_2, b_2] \subseteq [a_1, b_1]$$

Continuing in this fashion, we pick a subsequence (x_{n_k}) of (x_n) and (a_n) incr. (b_n) decr. such that

$$a_1 \le a_2 \le a_3 \ldots$$

$$b_1 \ge b_2 \ge b_3 \ldots$$

$$b_n - a_n = \frac{b_1 - a_1}{2^n} \longrightarrow 0$$

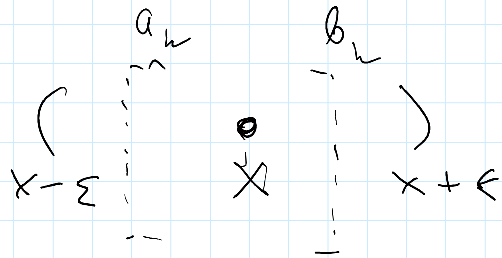
and $x_{n_k} \in [a_k, b_k]$

By NIP, $\bigcap_{n=1}^{\infty} [a_n, b_n]$ consists of 1 point, call it x

Claim $(x_{n_k}) \longrightarrow x$

of ∃ point, call it X

<u>Claim</u> $(x_{n_k}) \longrightarrow X$



Consequently $x_{n_k} \in (X-\varepsilon, X+\varepsilon)$

If I take $k^* > k$

then $x_{n_{k^*}} \in [a_{k^*}, b_{k^*}] \subset$

$[a_k, b_k] \subset (X-\varepsilon, X+\varepsilon)$

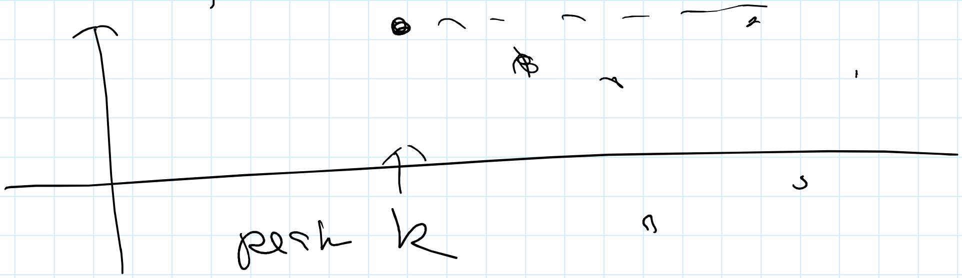
<u>Second proof</u>

<u>Theorem</u> Every sequence contains a decreasing subsequence or an increasing subsequence.

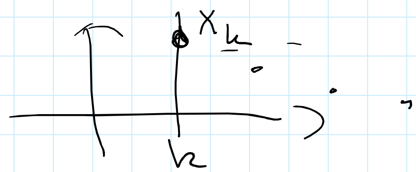
<u>pf of BW</u> given a <u>bounded</u> sequence, the Theorem produces a monotone subsequence. By MCT such a subsequence converges.

'Monotone' means 'increasing or decreasing'

Pf of theorem. Given a sequence (x_k) we say the sequence **peaks at k** if $k' > k \Rightarrow x_{k'} < x_k$



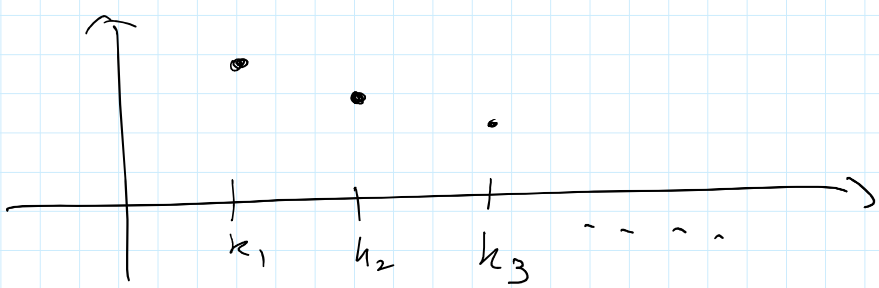
Example: if (x_k) decreasing, every $k \in \mathbb{N}$ will be a peak



if (x_k) is increasing, (x_k) does not peak at all.

Q $\rightarrow$ How many peaks are there?

Case 1: (x_k) has infinitely many peaks



x_{k_1} is a peak

$k_2 > k_1$ so $x_{k_2} < x_{k_1}$

and k_2 is a peak

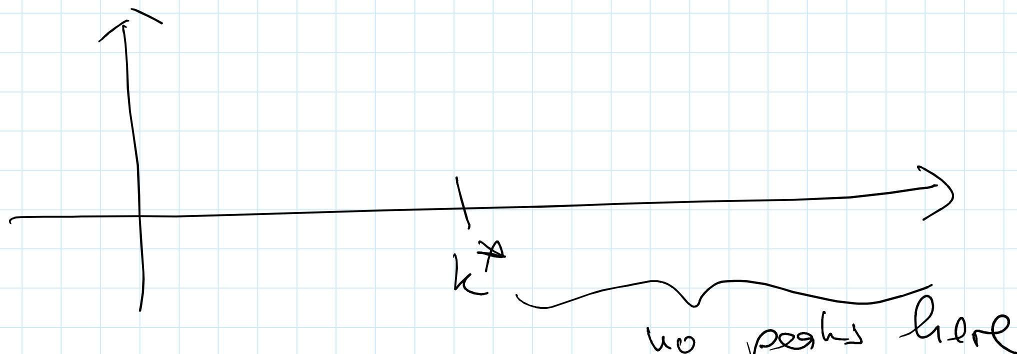
$k_3 > k_2$ so $x_{k_3} < x_{k_2}$

and k_3 is a peak

Consequently these peaks form a decreasing subsequence of the original sequence

<u>Case 2</u> (x_n) has only finitely many peaks

Say the biggest k at which the sequence peaks is k^*.



$k_1 = k^* + 1$ is not a peak

there is a $k_2 > k_1$ such that

$$x_{k_2} \geq x_{k_1}$$

at k_2 there is no peak either

so we can find

$$k_3 > k_2$$

with $x_{k_3} \geq x_{k_2}$ and so on

Consequently (x_{k_n}) will form an increasing subsequence of (x_n)

Definition (x_n) is a Cauchy sequence,
if for all $\varepsilon > 0$ $\exists N \in \mathbb{N}$ such that
$n, m > N$ then $|x_n - x_m| < \varepsilon$

Theorem Every Cauchy sequence converges

Step 1 Every convergent sequence is
a Cauchy sequence (is Cauchy)

pf Let $\varepsilon > 0$ be given
Since (x_n) is convergent, say with
limit x, there is $N \in \mathbb{N}$
s.t. if $n \geq N$
$|x_n - x| < \varepsilon/2$
now let $n, m \geq N$
then

$$|x_n - x_m| \leq |(x_n - x) + (x - x_m)|$$
$$\leq |x_n - x| + |x_m - x| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

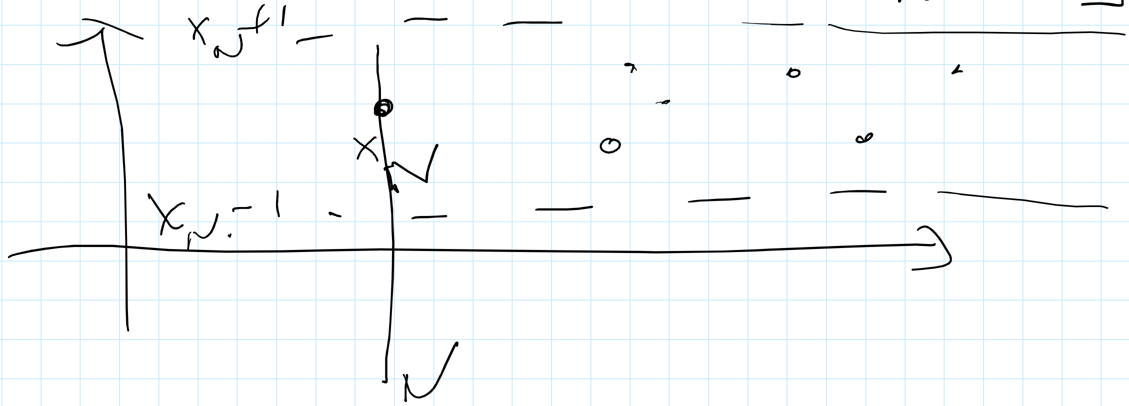
Step 2 Every Cauchy sequence
is bounded.
Let $\varepsilon = 1$ be given. Pick $N \in \mathbb{N}$
so that $n, m \geq N$
then $|x_n - x_m| < 1$
in particular if $n \geq N$
$|x_n - x_N| < 1$

In particular if $n \geq N$

$$|x_n - x_N| < 1$$

So for $n \geq N$

$$x_n \in [x_N - 1, x_N + 1]$$



for $\varepsilon = 1$

The rest of the proof is identical to the previous proof for convergent sequence.

Step 3 Consequence

A Cauchy sequence has a converging subsequence (by BW)

Next Time

Step 4 If a Cauchy sequence (x_n) has a converging subsequence (x_{n_k}) with limit x, then (x_n) converges to x