0916

Tuesday, September 16, 2025 14:45

Test 1 will be on Thursday Sept 25.

Last time BW Theorem

Every bounded sequence has a converging subsequence

Def (x_n) is Cauchy if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.that $\forall n, m \geq N$

$$|x_n - x_m| < \varepsilon$$

Cauchy-criterion Theorem (x_n) is Cauchy $\Leftrightarrow$ (x_n) is convergent to some $x \in \mathbb{R}$

Last time "$\Leftarrow$"

"$\Rightarrow$" Step 1 (x_n) Cauchy $\Rightarrow (x_n)$ bounded

Step 2 By ~~BW theorem~~ ~~(x_n)~~ has a converging subsequence, say (x_{n_k}) with limit x.

Step 3 (x_n) itself will then converge to x.

pf Let $\varepsilon > 0$.

Since (x_n) is Cauchy and (x_{n_k}) cv. to x

$\exists N \ \forall m, n \geq N, \ n_k \geq N$

(1) $|x_m - x_n| < \varepsilon/2$

(2) $|x_{n_k} - x| < \varepsilon/2$

for $n \geq N$ then $|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x|$

for a suitable $\leq |x_n - x_{n_k}| + |x_{n_k} - x|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

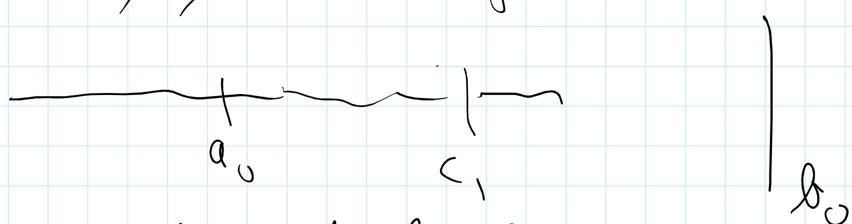
for a suitable choice of $n_k \ge N$ $\quad \le |x_n - x_{n_k}| + |x_{n_k} - x|$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(by (1)) $\qquad$ (by (2))

<u>Theorem</u> CC (+ AP) $\Rightarrow$ CA

<u>pf</u> let $A \neq \emptyset$, bounded from above



let $a_0 \in A$ and b_0 be an upper bound for A

let $c_1 = \dfrac{a_0 + b_0}{2}$

<u>Case 1</u> c_1 is an upper bound for A.

set $a_1 = a_0$ and $b_1 = c_1$

<u>Case 2</u> If c_1 is not u.b. of A

can find $a_1 \in A$, $a_1 \ge c_1$

and set $b_1 = b_0$

Continuing in this fashion we obtain an increasing (a_n)

$a_n \le b_n$, $\quad b_n - a_n \to 0$

(AP)

this implies that

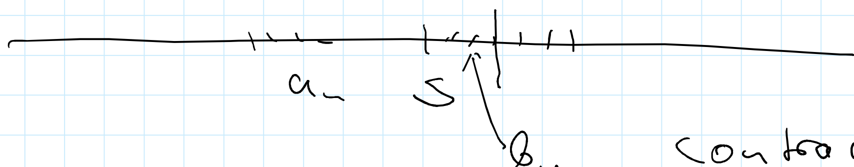
(a_n) is a Cauchy sequence

By CC (a_n) converges to some $s \in \mathbb{R}$

<u>Claim</u> s is $\sup A$

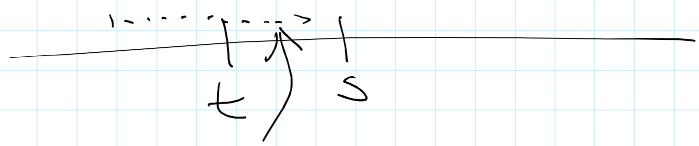
1. s is an upper bound of A

Suppose not $a \in A$



contradiction!

2. Suppose $t < s$



$a_n \in A$

thus t is not an upper bound of A

Concerning the use of the A P:

$$\ell = b_0 - a_0$$

$$b_1 - a_1 = \frac{1}{2}(b_0 - a_0) = \frac{\ell}{2}$$

$$b_2 - a_2 = \frac{\ell}{2^2}$$

$\vdots$

I have to know that $\frac{\ell}{2^k} \to 0$

$\neg AP$

CA $\Longleftrightarrow$ MCT $\Longleftrightarrow$ CC

ORDER of $\mathbb{R}$ $\downarrow$ lattice ordered structures

ORDER of $\mathbb{R}$ + DISTANCE

DISTANCE $\downarrow$ $\mathbb{R}^n$ (no order,

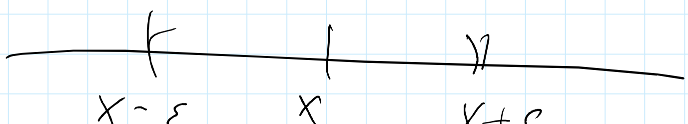
(Dr. Wojciechowski)

(norm is distance)

metric space (set + distance)

Open and closed sets

Def Let $x \in \mathbb{R}$, $\varepsilon > 0$. We write $V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$ "the ε-neighborhood of x"

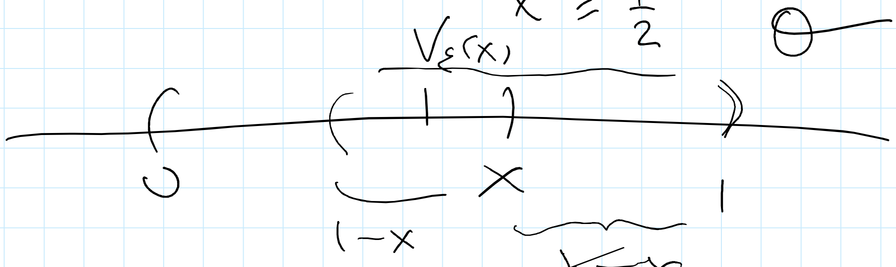


Def We say a set $\mathcal{O}$ is <u>open</u> if for every $x \in \mathcal{O}$ there is an $\varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq \mathcal{O}$.

Exa Open intervals are open

Ex $\mathcal{O} = (0,1)$

let $x \in \mathcal{O}$. assume W.l.o.g $x \geq \frac{1}{2}$

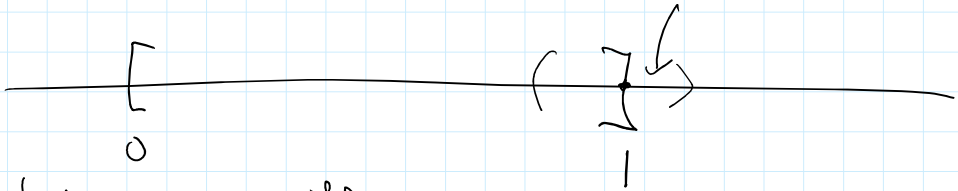


let $\varepsilon = 1 - x > 0$

$\ldots > 0$

then $V_\varepsilon(x) \subseteq \mathcal{O}$

Exa 2 $[0,1]$ is not open



Let $x = 1$. If $[0,1]$ was open, there would be $\varepsilon > 0$ s.that $V_\varepsilon(1) \subseteq [0,1]$

But $y = 1 + \frac{\varepsilon}{2}$

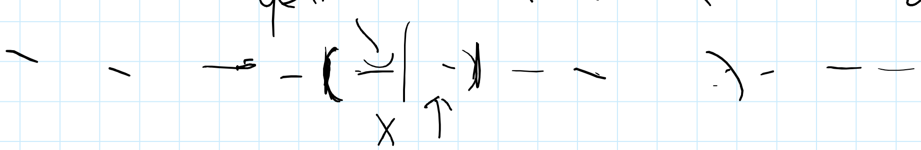
$y \in V_\varepsilon(1)$, but $y \notin [0,1]$

Exa 3 $A = \mathbb{R}$ is open

Exa $A = \mathbb{Q}$ is NOT open:

it does not contain any open interval

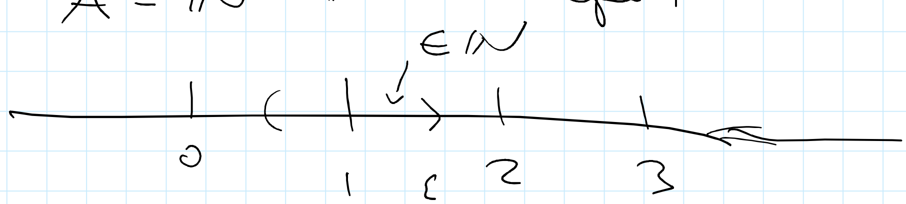
$y \in \mathbb{R} \setminus \mathbb{Q}$ (since $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$)



Exa 4 $A = \emptyset$ is open

(nothing to check)

Exa 5 $A = \mathbb{N}$ is NOT open



Theorem 1) The union of open sets is open

Pf Let $\mathcal{O}_\alpha$, $\alpha \in A$, be a collection of open sets collection of open sets and let $x \in \bigcup_{\alpha \in A} \mathcal{O}_\alpha$.

Then there is an $\alpha \in A$ with $x \in \mathcal{O}_\alpha$. Since $\mathcal{O}_\alpha$ is open can find $x \in V_\varepsilon(x) \subseteq \mathcal{O}_\alpha \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$

2) The intersection of finitely many open sets is open

Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be open sets, and let $x \in \bigcap_{k=1}^{n} \mathcal{O}_k$.

Then for every k, $x \in \mathcal{O}_k$

Since $\mathcal{O}_k$ is open can find $\varepsilon_k > 0$ such that $V_{\varepsilon_k}(x) \subseteq \mathcal{O}_k$

Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$

then $V_\varepsilon(x) \subseteq V_{\varepsilon_k}(x) \subseteq \mathcal{O}_k$ for all k

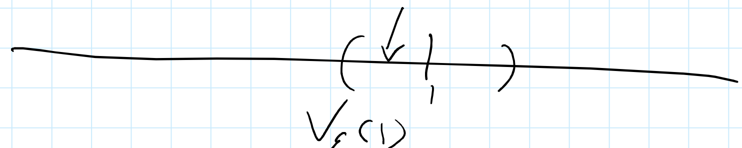
so $V_\varepsilon(x) \subseteq \bigcap_{k=1}^{n} \mathcal{O}_k$

Def $x \in \mathbb{R}$ is an accumulation point (limit point) of a set A if every neighborhood of x contains elements of A other than x.

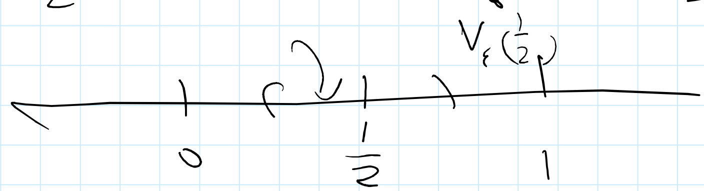
Exa $[0, 1]$ 1 is an acc. pt of $[0,1]$

Exa $[0,1]$ $\quad$ 1 is an acc. pt of $[0,1]$

For any $\varepsilon > 0$, $V_\varepsilon(1)$ contains pts in $[0,1]$ other than 1

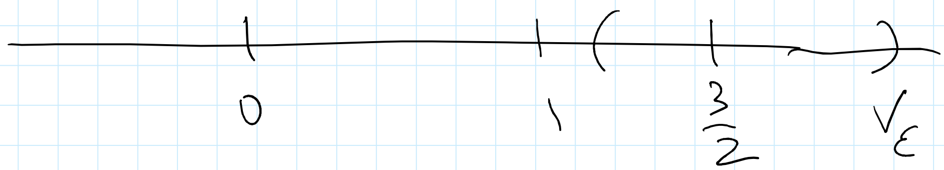


$\frac{1}{2}$ is an acc. pt of $[0,1]$



$\frac{3}{2}$ is not an acc. pt of $[0,1]$

$\varepsilon = \frac{1}{3}$



$V_\varepsilon(\frac{3}{2})$ contains no points in $[0,1]$

i.e. all acc. pts of $[0,1]$ are contained in $[0,1]$

Def A set F is closed if it contains all of its acc. pts.

$[0,1]$ is closed.

What about $(0,1)$? $(0,1)$ is not closed

$1 \notin (0,1)$, 1 is an acc. pt of $(0,1)$