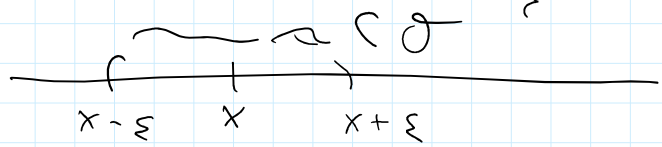
Last time $V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$

ε - nbhd of x

Def $\mathcal{O}$ is open if $\forall x \in \mathcal{O}\ \exists \varepsilon > 0$ s. that $V_\varepsilon(x) \subset \mathcal{O}$



examples open intervals, $\mathbb{R}$, $\emptyset$

not open $\mathbb{Q}$, finite sets

Aside Let X be a set.

Let $\mathcal{T} \subseteq \mathcal{P}(X)$

if 1) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$

2) arbitrary unions of elements in $\mathcal{T}$ are in $\mathcal{T}$

3) finite intersections of element in $\mathcal{T}$ are in $\mathcal{T}$

We call $(X, \mathcal{T})$ a topological space

The elements of $\mathcal{T}$ are called "open sets"

Can still do some analysis

Def $x_n \to x \iff \forall \mathcal{O} \in \mathcal{T}$ with $x \in \mathcal{O}$

there is an $N \in \mathbb{N}$ s. that

$\forall n \geq N: \ x_n \in \mathcal{O}$

Can't define what a Cauchy sequence is

Can't define what a Cauchy sequence is

Closed sets

Def Let $x \in \mathbb{R}$. x is acc. pt of a set A (limit pt.) if for every $\varepsilon > 0$

$V_\varepsilon(x) \cap A$ contains an element other than x.

(i.e. $V_\varepsilon(x) \cap A \setminus \{x\} \neq \emptyset$)

Def F is closed if it contains all of its accumulation points

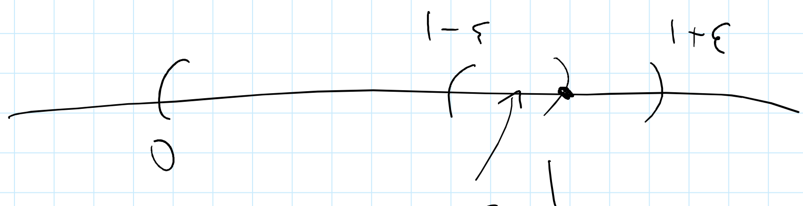
Exa $(0,1)$ is not closed:

1 is an acc. pt of $(0,1)$

Why Let $\varepsilon > 0$. Consider $(1-\varepsilon, 1+\varepsilon)$

$= V_\varepsilon(1)$. $V_\varepsilon(1) \ni 1 - \frac{\varepsilon}{2}$

$1 - \frac{\varepsilon}{2} \in V_\varepsilon(1) \cap (0,1)$



$1 - \frac{\varepsilon}{2} \in (0,1) \cap V_\varepsilon(1)$

$1 - \frac{\varepsilon}{2} \neq 1$

Claim set of acc pts of $(0,1)$

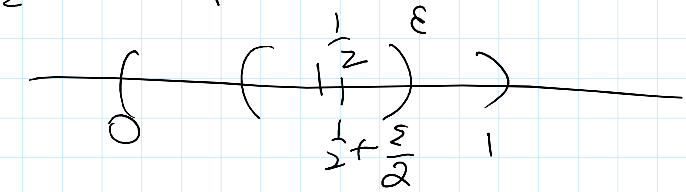
$= [0,1]$

0 acc pt of $(0,1)$

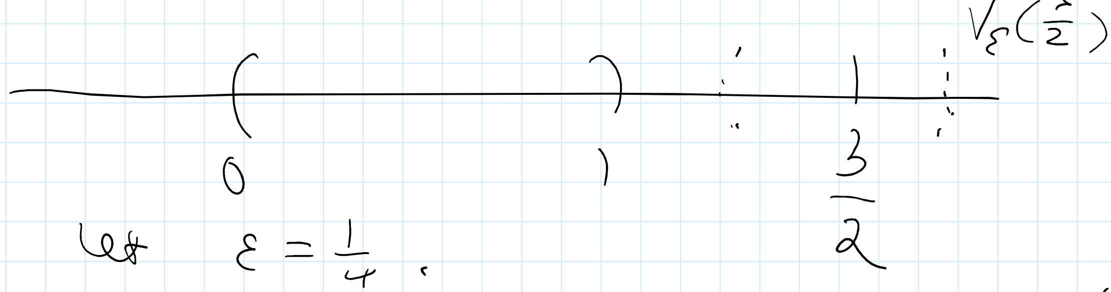
$[0,1]$

0 acc pt of $(0,1)$ ✓

$\frac{1}{2}$ acc. pt of $(0,1)$ ✓



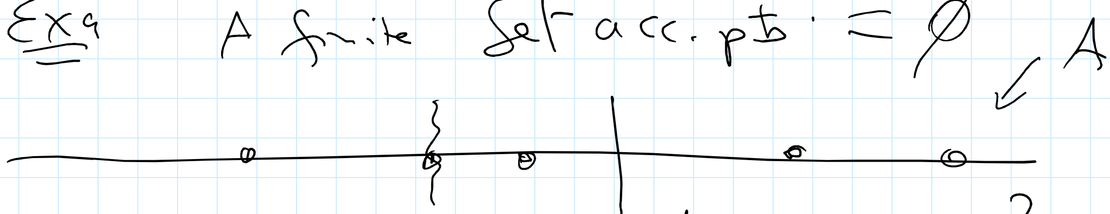
$\frac{3}{2}$ is not an acc pt of $(0,1)$



Let $\varepsilon = \frac{1}{4}$.

Then $V_\varepsilon(\frac{3}{2}) \cap (0,1) = \emptyset$

<u>Ex:</u> A finite set acc. pts $= \emptyset$



x^* is an acc. pt of A?

x is an acc. pt of A? NO

No: for small $\varepsilon > 0$

$V_\varepsilon(x^*) \cap A \setminus \{x^*\} = \emptyset$

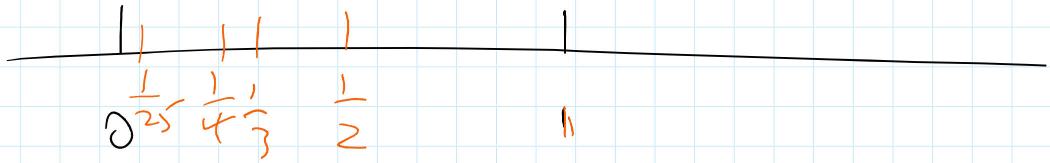
Consequence: finite sets are closed

Ex: $[0,1]$ set of acc pts: $[0,1]$

<u>closed</u>

Ex: $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

set of acc. pts :

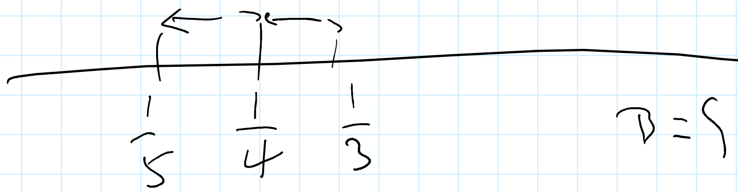


0 is the only acc. pt of A.

Arch. Prop.

$\varepsilon > 0$ arbitrary

If $x \neq 0$, x is not an acc pt of A



$B = \{0\} \cup A$ closed

Exs $\mathbb{N}$ no acc. pts

closed ✓

unbounded set without acc. pts

Exs $A = \emptyset$ closed

$A = \mathbb{R}$ set of acc. pts : $\mathbb{R}$

so $\mathbb{R}$ is closed.

! $\emptyset$ and $\mathbb{R}$ are also open.

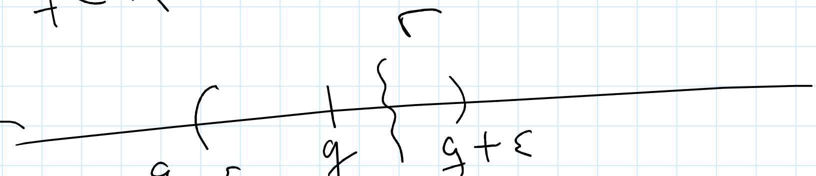
Exs neither closed nor open :

$A = [0, 1)$ not open "at 0"
not closed "at 1"

Exs $\mathbb{Q}$ set of acc pts:

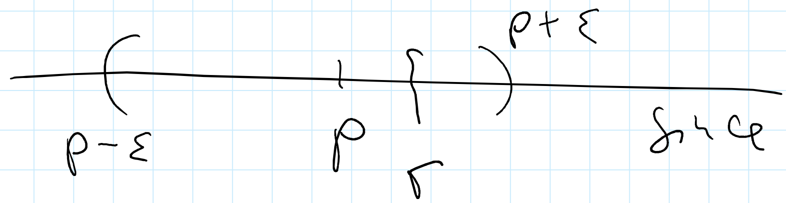
$q \in \mathbb{Q}$

let $\varepsilon > 0$ be given



$q < r < q + \varepsilon$ since $\mathbb{Q}$ is dense in $\mathbb{R}$

$p \in \mathbb{R} \setminus \mathbb{Q}$



since $\mathbb{Q}$ is dense in $\mathbb{R}$

there is a $r \in \mathbb{Q}$

$p < r < p + \varepsilon$

set of acc pts of $\mathbb{Q}$: $\mathbb{R}$

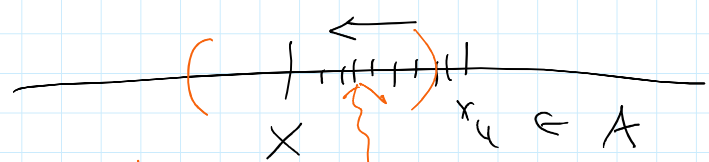
$\mathbb{Q}$ is not closed (not open either)

Thm $x \in \mathbb{R}$ is an acc pt of A

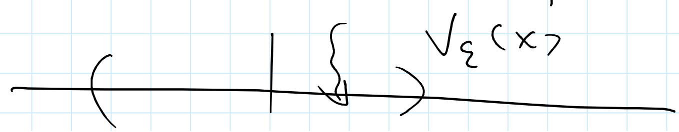
$\iff$ there is a sequence (x_n) in A
($x_n \neq x$ for all $n \in \mathbb{N}$) s.t. $x_n \longrightarrow x$.

Ex $\{1\}$ does not have an acc pt.

pf "$\Leftarrow$"



"$\Rightarrow$" $\in A$ for any $\varepsilon > 0$



x a cc. pt of A

For $\varepsilon = \frac{1}{n}$, $V_{\frac{1}{n}}(x) \cap A \setminus \{x\} \neq \emptyset$,
so we can pick $x_n \in A$
with $|x_n - x| < \frac{1}{n}$

Check that such a sequence converges to x ✓

<u>Thm</u> F is closed iff every Cauchy sequence in F converges to an element in F.

"local version of Cauchy criterion"

<u>Pf</u> "$\Rightarrow$" Suppose F is closed and (x_n) is a Cauchy sequence in F
(x_n) converges (by CC) to some $x \in \mathbb{R}$
We have to show $x \in F$

<u>Case 1</u> $x_n \neq x$ for all $n \in \mathbb{N}$
then x is an acc pt of F,
F is closed, so $x \in F$.

Case 1' There is a subseq (x_{n_k}) of (x_n) with $x_{n_k} \neq x$ for all k.
(x_n) cv. to x, so does (x_{n_k}). Thus ~~x is~~ an acc pt of ~~F~~ and thus ~~∈ F~~

<u>Case 2</u> There is a subsequence (x_{n_k}) of (x_n) with $x_{n_k} = x$

$x_{n_k} = x$ for all $k \in \mathbb{N}$.

i.e. (x_{n_k}) is a constant sequence of elements in F

so $(x_{n_k}) \to x \in F$.

$\Leftarrow$ Suppose x is an acc pt of F.

By previous theorem I can find a sequence (x_n) of elements in F $x_n \neq x$ $\forall n$, that converges to x

such a sequence is a Cauchy sequence. My hypothesis states that such a sequence converges to an element in F

Since limits are unique, (x_n) converges to $x \in F$.