09/23

Sunday, September 21, 2025 15:16

bbb

## Def. of a Cauchy Sequence

$(x_n)$ is called a Cauchy sequence if for all $\varepsilon > 0$ there is a $N \in \mathbb{N}$ s. that for all $m, n \geq N$:

$$|x_n - x_m| < \varepsilon$$

## If $(x_n) \to x$ then $\sqrt{x_n} \to \sqrt{x}$

$\sqrt{x} \neq 0$

Aside

Trick: $(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x}) = x_n - x$

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x} + \sqrt{x_n}} < \varepsilon \;?$$

$$\frac{|x_n - x|}{\sqrt{x} + \sqrt{x_n}} \leq \frac{|x_n - x|}{\sqrt{x}} < \varepsilon$$

$$|x_n - x| < \varepsilon \sqrt{x}$$

$\sqrt{x} \neq 0$

given $\varepsilon > 0$, I can find $N \in \mathbb{N}$ s. that $\forall n \geq N$

$$|x_n - x| < \varepsilon \sqrt{x}$$

then for such an $n \geq N$

$$|\sqrt{x} - \sqrt{x_n}| \leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\varepsilon \sqrt{x}}{\sqrt{x}} = \varepsilon$$

✓

Check yourself what happens if $x = 0$ !

## Find a converging subsequence from a non-converging bounded sequence

Don't know in general. But b. BW

Don't know in general, but by BW theorem there is such a converging subsequence.

Ex: $x_{2n} = 1$

$x_{2n-1} = \frac{1}{2n-1}$

$a_n = (-1)^n$

$(a_n)$ divergent since it has subsequences with different limits:

$a_{2n} = 1 \quad (a_{2n}) \to 1$

$a_{2n-1} = -1 \quad (a_{2n-1}) \to -1$

Example of a Cauchy sequ and explain why it converges

Theorem $(x_n)$ is a Cauchy sequence if and only if $(x_n)$ converges

Example $x_n = (1+\frac{1}{n})^n$

Euler showed that $(x_n)$ converges

He called the limit $e$

Can't show this directly

$$\left| (1+\tfrac{1}{n})^n - e \right| < ?$$

Show: if $(a_n) \to a$ & $(b_n) \to b$ then $a_n + b_n \to a + b$

**Pf** Let $\varepsilon > 0$ be given.

Choose $N \in \mathbb{N}$ s.t. that

$$\left.\begin{array}{l} |a_n - a| < \frac{\varepsilon}{2} \\ \text{and } |b_n - b| < \frac{\varepsilon}{2} \end{array}\right\} \text{ for all } n \geq N$$

Let $n \geq N$: Using the triangle inequality I obtain

$$\begin{aligned} &|(a_n + b_n) - (a + b)| \\ &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Every Cauchy sequence converges

1) Cauchy sequences are bounded
2) Use: BW-Theorem
   the Cauchy sequence has a converging subsequence, with limit, say, $x$.
3) Show: $(x_n)$ itself converges to $x$.

Given a concrete sequence $(x_n)$ show it converges to a limit $x$

__example__ $x_n = \dfrac{2n+1}{n+4}$

Will show: $(x_n)$ converges to 2

Aside

$$\left| \frac{2n+1}{n+4} - 2 \right|$$

$$= \left| \frac{2n+1}{n+4} - \frac{2n+8}{n+4} \right|$$

$$= \left| \frac{2n+1}{n+4} - \frac{2n+8}{n+4} \right|$$

$$= \frac{7}{n+4} \overset{!}{<} \varepsilon$$

$\iff 7 < (n+4)\varepsilon = n\varepsilon + 4\varepsilon$

$\iff 7 - 4\varepsilon < n\varepsilon$

$\iff \frac{7}{\varepsilon} - 4 < n$

Let $\varepsilon > 0$ be given.

Choose $N \in \mathbb{N}$ to exceed $\frac{7}{\varepsilon} - 4$

possible by Archim. principle

then for $n \geq N$ we obtain

$$\left| \frac{2n+1}{n+4} - 2 \right| \leq \frac{7}{n+4} \leq \frac{7}{N+4} < \varepsilon$$

## Completeness Axiom

Every bounded, $\neq \emptyset$, has a supremum

**Def** $s$ is the supremum of $A$

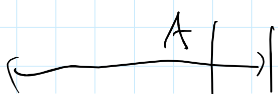
if (1) $s$ is an upper bound of $A$

(2) If $t$ is also an upper bound of $A$ then $t \geq s$.

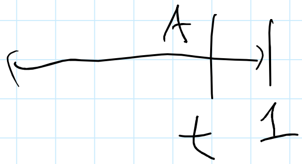
**Ex** $A = (0,1)$ $s = 1$

Why? (1) $s = 1 \geq x \ \forall x \in (0,1)$

(2) Suppose $t$ is an upper bound and $t < 1$



suppose ...
and $t < 1$



False since $\frac{1+t}{2} < 1$

and thus in $A$

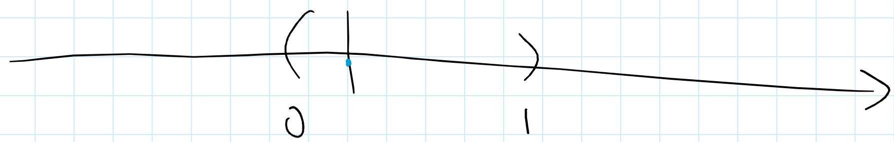
$\frac{1+t}{2} > t$

**Ex:** bounded set, no minimum, but infimum.

$A = (0,1)$ has infimum $0$, $\notin A$ $(\inf = 0)$

but no minimum!

$m = \min A$



if $m = \min A \in A$

$m > 0$

Can't be: $\underset{\substack{\cap \\ A}}{\frac{m}{2}} < m$

**Convergent $\Rightarrow$ Cauchy**

Let $(x_n)$ converge to $x$.

Let $\varepsilon > 0$ be given. Can find $N \in \mathbb{N}$ s. that

$$n \geq N \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$$

Now let $m, n \geq N$

then
$$|x_n - x_m|$$
$$= |x_n - x + x - x_m|$$
$$\leq |x_n - x| + |x_m - x|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Every sequence of real numbers has a monotone subsequence

Review "peak" !

HW 2.1

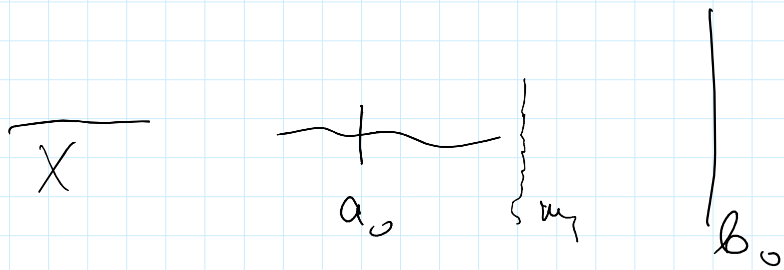
$$a_n = \sqrt{\frac{2n+5}{n+2}} \quad (\text{cv. to } \sqrt{2})$$

combine "$\sqrt{\ }$" proof with the "quotient" algebra

HW 1.5

$$NIP + AP \Rightarrow CA$$

Outline: let $X \neq \emptyset$, bounded from above.




Can find $a_0 \in X$, $b_0$ an upper bound of $X$

Now use bisection

Let $m_1 = \frac{a_0 + b_0}{2}$

— if $m_1$ is an upper bound for $X$

$a_1 = a_0 \quad b_1 = m_1$

— if not can find $a_1' \in X$, $a_1' \geq m_1$

$a_1 = a_1' \quad b_1 = b_0$

Continuing like this

$a_n \nearrow \qquad (b_n - a_n) \to 0$

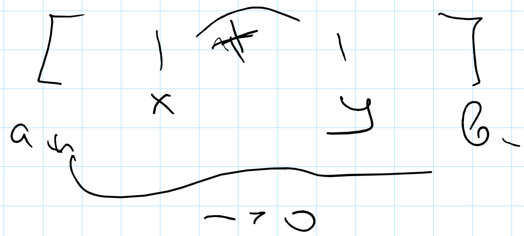
$b_n \searrow$

Therefore $[a_n, b_n]$ form a sequence of nested closed bounded intervals

By NIP $\bigcap_{n\in\mathbb{N}} [a_n, b_n] \neq \emptyset$

Since $b_n - a_n \to 0$, AP implies $\bigcap_{n\in\mathbb{N}} [a_n, b_n]$ consists of (at most 1 point)

$x, y \in \bigcap_{n\in\mathbb{N}} [a_n, b_n]$




So, $\bigcap_{n\in\mathbb{N}} [a_n, b_n] = \{s\}$

Check: $s$ is the sup of $X$

$\mathbb{Q}$ is sequentially dense in $\mathbb{R}$

i.e. for every $x \in \mathbb{R}$ there is $(q_n) \in \mathbb{Q}$, $q_n \to x$

Let $n \in \mathbb{N}$ be given

$(x - \frac{1}{n}, x + \frac{1}{n})$ contains a rational number, call it $q_n$, since $\mathbb{Q}$ is dense in $\mathbb{R}$

Check: $(q_n) \to x$