P1

$$a_n = \frac{n^2}{n^2+1} \longrightarrow 1$$

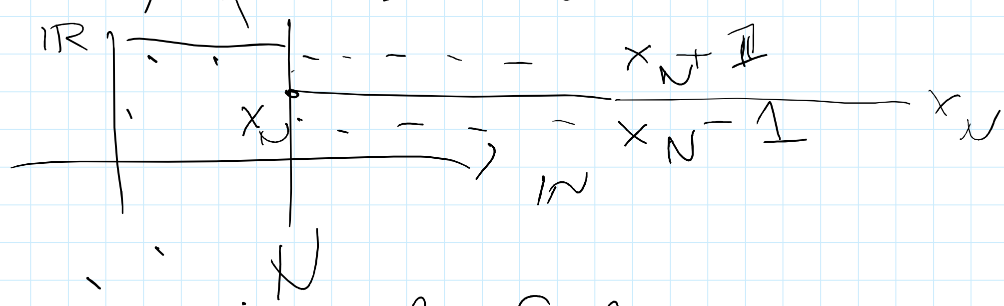
aside
$$\left|\frac{n^2}{n^2+1} - 1\right| = \left|\frac{n^2}{n^2+1} - \frac{n^2+1}{n^2+1}\right| = \left|\frac{-1}{n^2+1}\right| = \frac{1}{n^2+1} \le \frac{1}{n^2} \overset{!}{<} \varepsilon$$

$$n > \frac{1}{\sqrt{\varepsilon}}$$

Pf Let $\varepsilon > 0$ be given. Pick $N \in \mathbb{N}$ s. that $N > \frac{1}{\sqrt{\varepsilon}}$. Then for any $n \ge N$

$$|a_n - 1| = \frac{1}{n^2+1} < \frac{1}{n^2} \le \frac{1}{N^2} < \varepsilon.$$

P2 Cauchy sequences are bounded.



Let (x_n) be a Cauchy sequence

Pf For $\varepsilon = 1$ can find $N \in \mathbb{N}$ s. that for any $m, n \ge N$

$$|x_n - x_m| < \varepsilon = 1$$

i.e. for $m = N$ we obtain

$$|x_n - x_N| < 1$$

i.e. $|x_n| < |x_N| + 1$

let $M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|\}$

then for all $n \in \mathbb{N}$

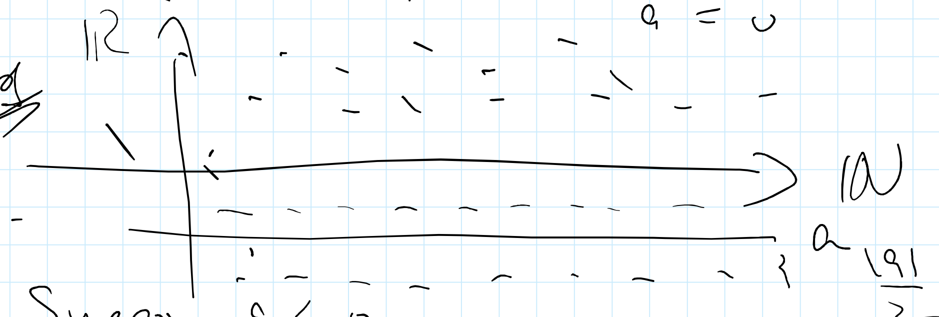
$$|x_n| \le \max\{M, |x_N| + 1\}$$

P3

$$\left.\begin{array}{l} a_n \to a \\ a_n \ge 0 \ \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow a \ge 0$$

N.B. false for ">": $a_n = \frac{1}{n} > 0$, $a = 0$

pf



Suppose $a < 0$

let $\varepsilon = \frac{|a|}{2}$. then there is an $N \in \mathbb{N}$ s. that for all $n \ge N$

$$|a_n - a| < \frac{|a|}{2},$$ a contradiction

P4

$$a_1 = 1 \qquad a_{n+1} = \sqrt{2a_n}$$

Show (a_n) converges

(Hint: part 1)

pf Idea: show (a_n) incr. & bdd then by MCT it converges

(a_n) increasing (& ind.)

$1 = a_1 \le a_2 = \sqrt{2}$ ✓

$n \to n+1$ we know $a_n \le a_{n+1}$

we need to know $a_{n+1} \le a_{n+2}$

$$a_{n+1} = \sqrt{2a_n} \le \sqrt{2a_{n+1}} = a_{n+2} \checkmark$$

(a_n) is bounded, say by 10

$a_1 = 1 \le 10 \checkmark$

$n \to n+1$ $\quad a_{n+1} = \sqrt{2 \cdot a_n} \le \sqrt{2 \cdot 10} = \sqrt{20}$

$\le 10 \checkmark$

P5 $\quad A \neq \emptyset$ bdd from above.

$s = \sup A$

show $s \in A$ or s is an acc pt of A.

pf: Assume $s \notin A$. ~~let $\varepsilon > 0$ be~~ given.

$s - \varepsilon$ is not an upper bd for A,

so there is an $a \in A$: $a > s - \varepsilon$

then $s \neq a \in V_\varepsilon(s)$.

Def $\quad x \in \mathbb{R}$ is an acc. pt (limit pt) of X

if $\forall \varepsilon > 0 \quad V_\varepsilon(x) \cap X \setminus \{x\} \neq \emptyset$

Thm $\quad x \in \mathbb{R}$ is an acc pt of X

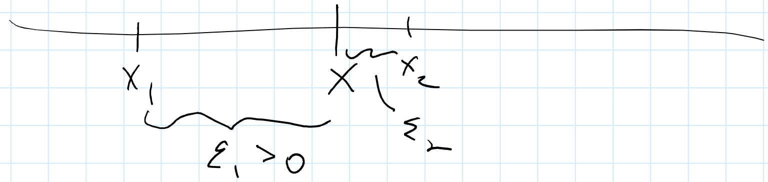
iff every nbhd of x contains

infinitely many elements of X.

$\Leftarrow$ is straightforward

$\Rightarrow$ let $\varepsilon > 0$ ~~be~~ given. let x_1 be in

$V_\varepsilon(x) \cap A, \quad x \neq x_1$



$V_{\varepsilon_1}(x) \cap A$ contains an element $x_2 \in A$

$x_2 \neq x$. Note $x_2 \neq x_1$.

Continuing in this fashion we obtain infinitely many distinct elements in $V_\varepsilon(x) \cap A$.

Def: F is closed if it contains all of its acc. pts.

Ex: $[0,1]$, $\emptyset$, $\mathbb{R}$, $\mathbb{N}$, $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$

Theorem F is closed iff its complement is open

$\Rightarrow$ Suppose F is closed and $\mathcal{O} = \mathbb{R} \setminus F$ is not open. Then there is an $x \in \mathcal{O}$ s. that $\forall \varepsilon > 0$ $V_\varepsilon(x) \not\subseteq \mathcal{O}$ $(x \notin F)$

This implies $V_\varepsilon(x) \cap \mathbb{R} \setminus \mathcal{O} \neq \emptyset$

i.e. $V_\varepsilon(x) \cap F \neq \emptyset$

So x is an acc pt of F, a contradiction.

$\Leftarrow$ Suppose $\mathcal{O}$ is open, but its complement $F = \mathbb{R} \setminus \mathcal{O}$ is not closed. So there is an acc. pt x of F, $x \notin F$. So $x \in \mathcal{O}$ (open!)

So there is an $\varepsilon > 0$ s. that epsilon

$\in$ Σ

φ ϕ

θ ϑ

$V_\varepsilon(x) \subseteq \mathcal{O}$

i.e. $V_\varepsilon(x) \cap \mathbb{R}\setminus\mathcal{O} = F = \emptyset$,

contradicting that x is an acc pt of F.

<u>Corollary</u> Arbitrary intersections of closed sets are closed; finite unions of closed sets are closed.

<u>Pf</u> previous result about open sets and deMorgan's Laws:

$$\bigcap_\alpha \mathbb{R}\setminus F_\alpha = \mathbb{R}\setminus\Big(\bigcup F_\alpha\Big)$$

$$\bigcup_\alpha \mathbb{R}\setminus F_\alpha = \mathbb{R}\setminus\Big(\bigcap_\alpha F_\alpha\Big)$$

An <u>interesting set</u>:

<u>Def</u> Two sets A and B have the same cardinality if there is a bijection

$\varphi : A \to B$

(φ is 1-1 and onto)

We write $A \sim B$

<u>Remark</u> $\sim$ is an equivalence relation

refl $A \sim A$

symm. $A \sim B \Rightarrow B \sim A$

transitive $A \sim B \wedge B \sim C \Rightarrow A \sim C$

Thm (G.G.) $\mathbb{N} \sim 2\mathbb{N} \leftarrow$ even numbers

Pf Consider $\varphi(n) = 2n$

Fact Finite sets have the same cardinality iff both sets have the same number of elements

Thm $\mathbb{N} \sim \mathbb{Z}$

$\varphi(1) = 0,\ \varphi(2) = 1,\ \varphi(3) = -1$

$\varphi(4) = 2 \quad \varphi(5) = -2, \dots$

Theorem (Cantor) $\mathbb{N} \sim \mathbb{Q}$

Pf Show $\mathbb{N} \sim \mathbb{Q}^+$

$$
\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\
\frac{1}{2} & \cancel{\frac{2}{2}} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \cdots & & & \\
\vdots & & & & &
\end{array}
$$

$\varphi(1) = \frac{1}{1} \quad \varphi(2) = \frac{2}{1} \quad \varphi(4) = \frac{3}{1}$

$\varphi(3) = \frac{1}{2} \quad \varphi(5) \ \frac{1}{3}$

$\varphi(6) = \frac{4}{1}$

$\varphi(7) = \frac{3}{2}$

$\varphi(8) = \frac{2}{3}$

$\varphi(9) = \frac{1}{4}$

Check φ is a bijection

$\varphi(9) = \frac{1}{4}$

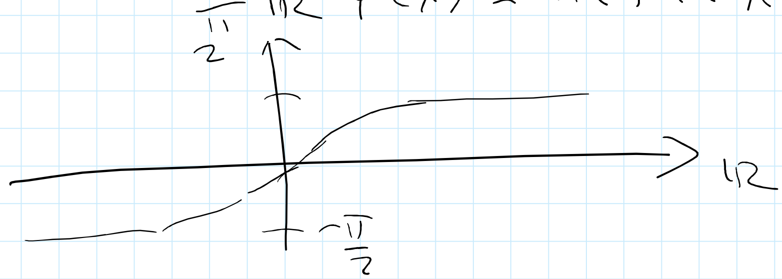
Q Do all infinite sets have the same cardinality as $\mathbb{N}$?

NO $\mathbb{N} \not\sim \mathbb{R}$

Preparation:

$\mathbb{R} \sim \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\mathbb{R}$ $\varphi(x) = \arctan x$ is a bijection



$\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \sim (0,1)$ exercise

Next time | $(0,1) \sim \mathcal{P}(\mathbb{N})$ ← power set

$\mathbb{N} \not\sim \mathcal{P}(\mathbb{N})$