Cardinality $X \sim Y$ if there is a bijection
$\varphi : X \to Y$

If $X \sim \mathbb{N}$, we say X is countable
(countably infinite)

($X \preceq Y$ if there is a 1-1 fct.
$\varphi : X \to Y$)

Cantor-Bernstein-lemma:
If $X \preceq Y$ and $Y \preceq X$
$\Rightarrow X \sim Y$

$X \sim Y$ defines an equivalence relation.

Last time: $\mathbb{N} \sim 2\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

Georg Cantor (2nd half of 19th century)

Today $\mathbb{R} \nsim \mathbb{N}$

Last time: $\mathbb{R} \sim (-\frac{\pi}{2}, \frac{\pi}{2}) \sim (0,1)$

2 steps: ① $\mathbb{R} \sim \mathcal{P}(\mathbb{N}) \leftarrow$ power set of $\mathbb{N}$

② For any set: $X \nsim \mathcal{P}(X)$

① Outline We can write every $x \in (0,1)$
in its binary expansion:
$x = 0.d_1 d_2 d_3 \ldots$.
$d_i \in \{0,1\}$

$$x = \sum_{i=1}^{\infty} \frac{d_i}{2^i}$$

Define $\varphi : (0,1) \to \mathcal{P}(\mathbb{N})$
$(0.d_1 d_2 d_3 \ldots) \stackrel{\varphi}{\longmapsto} \{ n \in \mathbb{N} / d_n = 1 \}$

$\cdot$ Define $\varphi : (0,1) \to \mathcal{P}(\mathbb{N})$

$(0.d_1d_2d_3\ldots)_2 \xmapsto{\varphi} \{n \in \mathbb{N} / d_n = 1\}$

Examples

$\frac{1}{2} = (0.1)_2 \xmapsto{\varphi} \{1\}$

$\frac{3}{4} = (0.11)_2 \xmapsto{\varphi} \{1,2\}$

$(0.101010\ldots)_2 \xmapsto{\varphi} \{1,3,5,7,\ldots\}$

$(0.0\overline{1001})_2 = (0.0100100\ldots)_2 \longleftarrow \{2,5,8,11,\ldots\}$

irrational!

$(0.10010001\ldots)_2$

φ is more or less a bijection.

Problem

Does every real number have a unique binary expansion?

$(0.5)_{10} = (0.49999\ldots)_{10} = (0.4\overline{9})_{10}$

$(0.1)_2 = (0.0\overline{1})_2$

this only happens when one of the representations is terminating

its cardinality is : $\sim \mathbb{N}$

② Cantor's Theorem

For any set : $X \not\sim \mathcal{P}(X)$

Pf: Let X be given

Suppose, to the contrary, that $X \sim \mathcal{P}(X)$

i.e. there is a bijection

$\varphi : X \to \mathcal{P}(X)$.

$B = \{x \in X \mid x \notin \varphi(x)\}$

$$B = \{ x \in X \mid x \notin \underbrace{\varphi(x)}_{\substack{\in \mathcal{P}(X) \\ \subseteq X}} \}$$

Since φ is onto, there is a $b \in X$ s. that $\varphi(b) = B$

Is $b \in B$? NO since if $b \in B$
$b \notin \varphi(b) = B$

Is $b \notin B$? NO since if $b \notin B$
then $b \in \varphi(b) = B$

contradiction!

There are infinitely many infinite cardinalities:

$$\mathbb{N} \not\sim \mathcal{P}(\mathbb{N}) \not\sim \mathcal{P}(\mathcal{P}(\mathbb{N})) \ldots$$

Continuum Hypothesis

$$\mathbb{N} \leq X \leq \mathbb{R} \Rightarrow X \sim \mathbb{N} \text{ or } X \sim \mathbb{R}$$

Cantor set

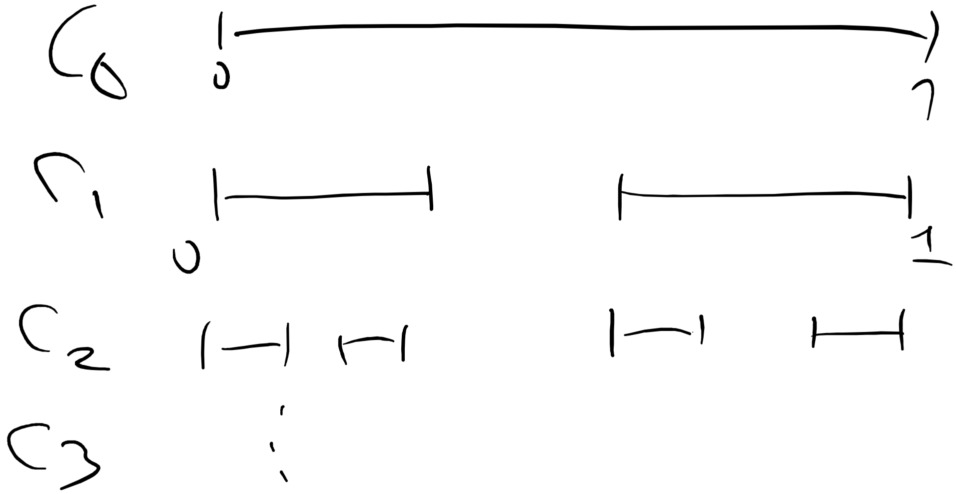
$$C_0 = [0, 1]$$

$$C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

$$C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$

$$C_3 = \ldots$$

$C_3 = \dots,$



$$C = \bigcap_{n \in \mathbb{N}} C_n$$

→ $C \neq \emptyset$: it contains the endpoints of all participating intervals.
C contains a countable set ✓

→ C is closed :
each C_n is closed as a finite union of closed intervals
C is closed as an intersection of the closed C_n's.

→ C does not contain any open interval

part of C_n

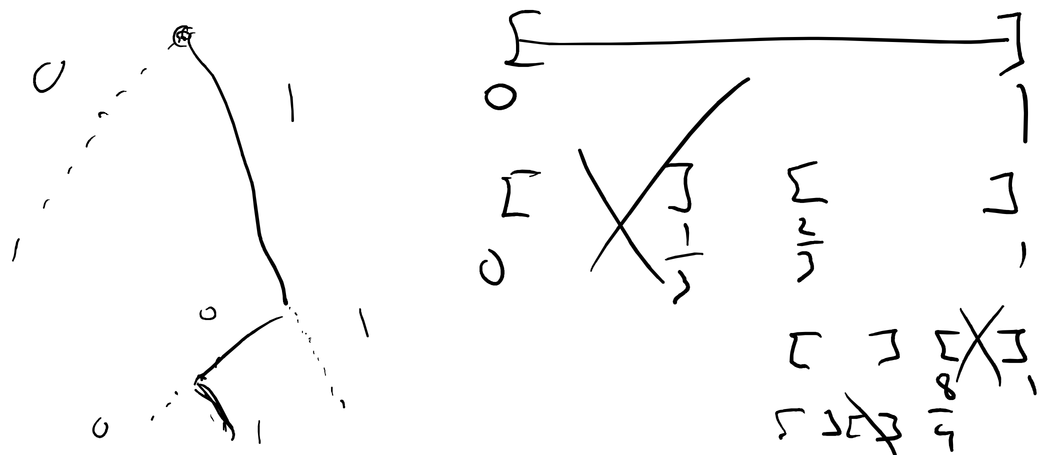
$O \not\subset C_n$ and thus $O \not\subset C$

→ C equals the set of its acc pts.
(such as $\emptyset$, $\mathbb{R}$, $[0,1]$) "perfect sets"

(such as $\emptyset$, $\mathbb{R}$, $[0,1]$)

→ C is uncountable: $C \sim \mathbb{R}$

How to describe elements in C?



$x \in C \longleftrightarrow 101\ldots$

$\updownarrow$

$\{1, 3, \ldots\} \; (\in \mathcal{P}(\mathbb{N}))$

→ C has "length" 0

$C_0 = [0,1]$ length: 1

$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$ length $\frac{2}{3}$

C_2 length $= \frac{4}{9}$

$\vdots$

C_n length $= (\frac{2}{3})^n$

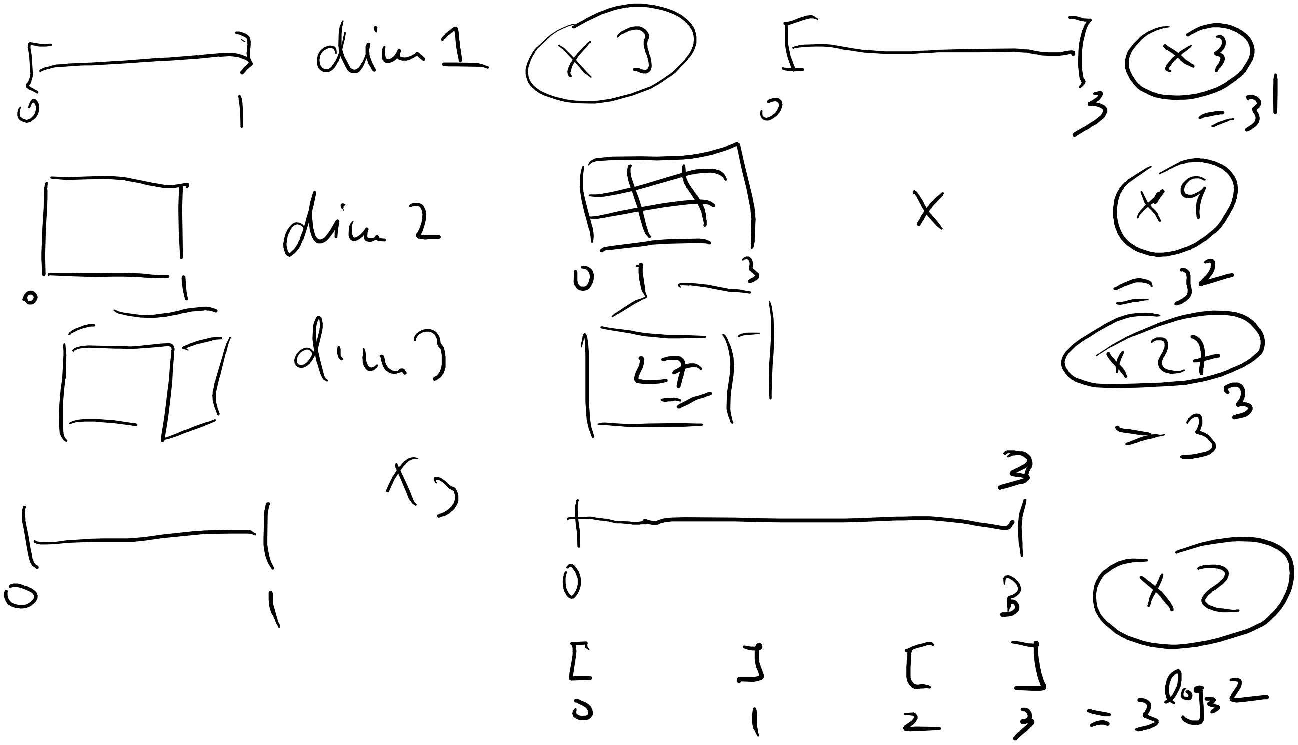
C length $= 0$

$(\frac{2}{3})^n \to 0$

C has "measure" 0

→ $\dim C = \log_3 2$

→ $\dim C = \log_3 2$



$\dim C = \log_3 2 \approx 0.71\ldots$

→ Hausdorff dimension,

Minkowski dimension

"Geometric Analysis"

<u>Compact sets</u>

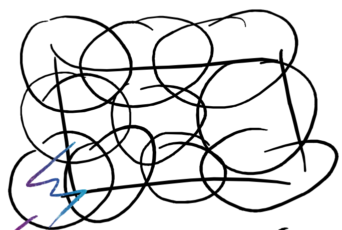
given a set $X \subset \mathbb{R}$,

we say that a collection of open sets $\{O_\alpha \mid \alpha \in A\}$

is an <u>open cover of X</u>

is an <u>open cover of X</u>

if $X \subset \bigcup_{\alpha \in A} \mathcal{O}_\alpha$



take out

We say $\{\mathcal{O}_1, \dots \mathcal{O}_n\}$ is a finite subcover of $\{\mathcal{O}_\alpha \mid \alpha \in A\}$

if $\mathcal{O}_k = \mathcal{O}_{\alpha_k}$ for some $\alpha_k \in A$

and $\bigcup_{k=1}^{n} \mathcal{O}_k \supset X$