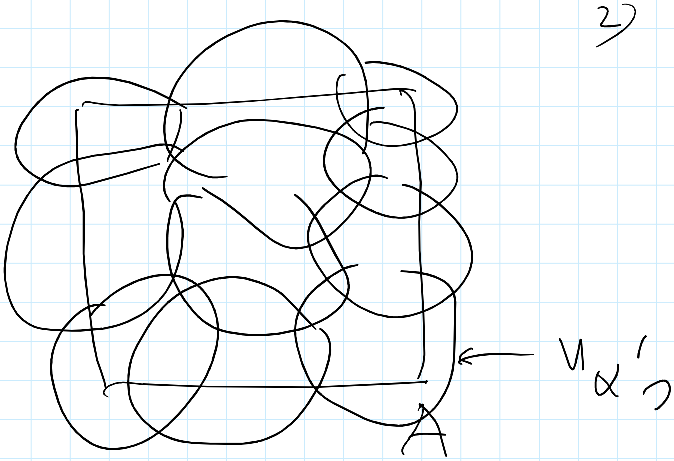
Compactness

Def given a set A we say

$$\mathcal{O} = \{U_\alpha \mid \alpha \in A\} \text{ is an open cover for } A$$

if 1) U_α open $\forall \alpha$

2) $\bigcup_{\alpha \in A} U_\alpha \supseteq A$



Def $\mathcal{O}$ is an open cover for A

then $\mathcal{O}^* = \{U_1, U_2, U_3, \dots, U_n\}$

is called a finite subcover of $\mathcal{O}$

if 1) $U_k \in \mathcal{O}$

2) $\bigcup_{k=1}^{n} U_k \supset A$

Definition A set is compact if any open cover for it contains a finite subcover.

Logical structure $\forall \mathcal{O}\ \exists \mathcal{O}^*$ → compact

$\exists \mathcal{O}\ \forall \cancel{\mathcal{O}^*}$ → not compact

Example Finite sets are compact.

Pf: let $X = \{x_1, \dots, x_n\}$

and let $\mathcal{O} = \{U_\alpha \mid \alpha \in A\}$ be an open cover for X

Thus $\bigcup_{\alpha \in A} U_\alpha \supset X$

For $x_i \in X$ there is a U_{α_i}, $\alpha_i \in A$ s.t. that $x_i \in U_{\alpha_i}$

thus $\mathcal{O}^* = \{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ will be a subcover of $\mathcal{O}$ for X

Example $(0,1)$ is not compact!

$U_n = (\frac{1}{n}, 1)$

$\mathcal{O} = \{U_n \mid n \in \mathbb{N}\}$

1) U_n open ✓

2) $\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1) = (0,1) \supseteq (0,1)$

$\subseteq$ ✓ $\supseteq$ requires Archim. Principle

$\mathcal{O}$ is therefore an open cover

Now consider

$U_1 = \emptyset$

$U_2 = (\frac{1}{2}, 1)$

$U_3 = (\frac{1}{3}, 1)$

$\vdots$

$\mathcal{O}^* = \{U_{n_1}, U_{n_2}, U_{n_3}, \dots, U_{n_k}\}$ be a finite collection of sets in $\mathcal{O}$

wlog $n_1 < n_2 < n_3 < \dots n_k$

$\bigcup_{i=1}^{k} U_{n_i} = U_{n_k} = (\frac{1}{n_k}, 1) \not\supset (0,1)$

Theorem X compact $\Rightarrow$ X is closed

Pf Suppose X is not closed, say x_0 is an acc pt of X

$U_n = \mathbb{R} \setminus [\dots] \quad x_0 \notin X$

Pf Suppose X is not closed, say x_0 is an acc pt of X, $x_0 \notin X$

$$U_n = \underbrace{\mathbb{R} \setminus \left[x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n}\right]}_{\text{open}}$$

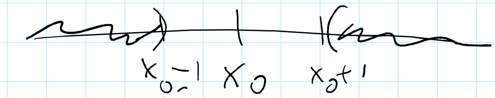
$$\mathcal{O} = \{ U_n \mid n \in \mathbb{N} \}$$

1) U_n open ✓

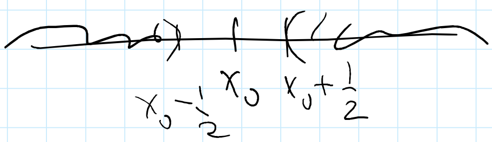
2) $\displaystyle\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \mathbb{R} \setminus \left[x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n}\right]$

$= \mathbb{R} \setminus \{x_0\} \supseteq X$ ✓

U_1



U_2

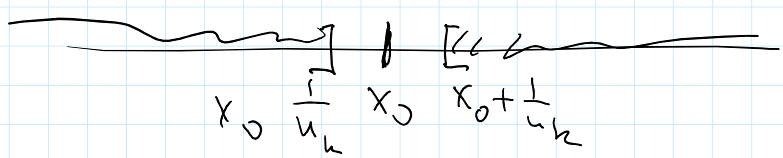


Now consider finite collection of sets in $\mathcal{O}$

$$\mathcal{O}^* = \{ U_{n_1}, U_{n_2}, \dots U_{n_k} \}$$

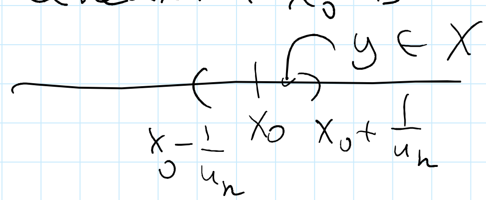
Wlog $n_1 < n_2 < \dots < n_k$

$$\bigcup_{i=1}^{k} U_{n_i} = U_{n_k} = \mathbb{R} \setminus \left[x_0 - \tfrac{1}{n_k}, x_0 + \tfrac{1}{n_k} \right]$$



$\displaystyle\bigcup_{i=1}^{k} U_{n_i} \not\supseteq X$

Remember: x_0 is an acc. pt of X



$\longrightarrow$ $\mathbb{N}$ is not compact

$U_n = (0, n)$

$\mathcal{O} = \{ U_n \mid n \in \mathbb{N} \}$

1) U_n open ✓

2) $\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} (0, n) = (0, \infty) \supseteq \mathbb{N}$

Now consider any finite subcollection of $\mathcal{O}$

$\mathcal{O}^* = \{ U_{n_1}, U_{n_2}, \ldots U_{n_k} \}$

WLOG $n_1 < n_2 < \ldots n_k$

then $\bigcup_{i=1}^{k} U_{n_i} = U_{n_k} = (0, n_k) \not\supseteq \mathbb{N}$

use Archie

Theorem X is compact $\Rightarrow$ X is Bounded

pf Suppose X is not Bounded

$\mathcal{O} = \{ U_n \mid n \in \mathbb{N} \}$

$U_n = (-n, n)$

1) U_n open ✓

2) $\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} (-n, n) = \mathbb{R} \supseteq X$

finite subcollection of $\mathcal{O}$

$\mathcal{O}^* = \{ U_{n_1}, U_{n_2}, \ldots U_{n_k} \}$ $\quad n_1 < n_2 < \ldots < n_k$

$\bigcup_{i=1}^{k} U_{n_i} = \bigcup_{i=1}^{k} (-n_i, n_i)$

$= (-n_k, n_k) \not\supseteq X$

since n_k is not an

since n_k is not an upper bound for $|x|$, $x \in X$

$\rightarrow$ $[0,1]$ is compact

let $\mathcal{O} = \{U_\alpha \mid \alpha \in A\}$ be an open cover for $[0,1]$ and suppose no finite subcollection will be a cover for $[0,1]$

$\rightsquigarrow$ condensation argument!

$a_0 = 0$ $\quad b_0 = 1$

$c_0 = \frac{1}{2}$

$[0, \frac{1}{2}]$ $\quad \mathcal{O}$ is also an open cover for $[0, \frac{1}{2}]$

Case 1 There is no finite subcover of $\mathcal{O}$ for $[0, \frac{1}{2}]$

set $a_1 = 0 \quad b_1 = \frac{1}{2}$

Case 2 there is a finite subcover of $\mathcal{O}$ for $[0, \frac{1}{2}]$

then there is no finite subcover of $\mathcal{O}$ for $[\frac{1}{2}, 1]$

in this case

set $a_1 = \frac{1}{2}$, $b_1 = 1$

$\rightarrow$ $\mathcal{O}$ does not have a finite subcover for $[a_1, b_1]$

Continue in this fashion:

$a_n \nearrow \quad b_n \searrow \quad b_n - a_n \longrightarrow 0$

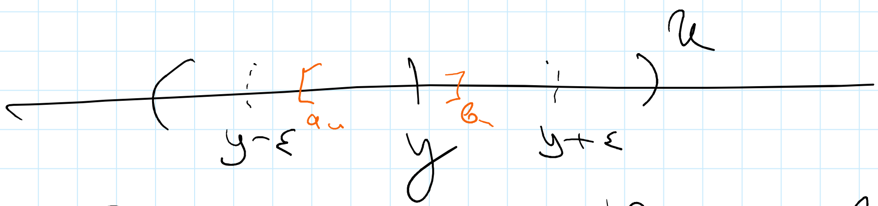
$\mathcal{O}$ does not a finite subcover for

$\mathcal{O}$ does not [have] a finite subcover for $[a_n, b_n]$

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{y\} \text{ for some } y \in \mathbb{R}$$

Is $y \in X$? Yes! $y \in [0,1] = X$

since it is an acc pt of $[0,1]$



Since $y \in [0,1]$ there is a $U \in \mathcal{O}$ with $y \in U$

For large enough n,

$$[a_n, b_n] \subset U$$

So I have found a one-element subcover of $\mathcal{O}$ that covers $[a_n, b_n]$ $\blacksquare$

Similarly $[a, b]$ is compact for any $a \leq b$.

Theorem Closed subsets of compact sets are compact.

Theorem X closed and bounded $\Rightarrow$ X is compact

pf X is bounded, so

$X \subset [a, b]$ for some a and b

Pf X is bounded, so

$X \subset [a,b]$ for some a and b

X is closed.

By previous theorem,

X is compact.

$$\boxed{\begin{array}{l}\text{Heine Borel Theorem} \\ X \text{ compact} \iff X \text{ closed and bounded}\end{array}}$$

"Compact sets are the natural generalization of finite sets!"

Pf (closed set $\subset$ cpt set is compact)

Let A be closed, $A \subseteq B$, B compact.

Let $\mathcal{O}$ be an open cover for A

(have to find a finite subcover)

Let $\hat{\mathcal{O}} = \mathcal{O} \cup \{ \underbrace{\mathbb{R} \setminus A}_{\text{open}} \}$ closed

1) all sets in $\tilde{\mathcal{O}}$ are open

2) $\bigcup_{U \in \tilde{\mathcal{O}}} U = \mathbb{R} \supseteq B$

B is compact, so $\tilde{\mathcal{O}}$ has a finite subcover $\tilde{\mathcal{O}}^*$ for B

but then $\mathcal{O}^* = \tilde{\mathcal{O}}^* \setminus \{\mathbb{R} \setminus A\}$

will be a finite subcover of $\mathcal{O}$ for A. $\square$

$\tilde{\sigma}$ for $'A$. $\blacksquare$