No office hours today!
Next: Monday 3–4

Last time: Heine–Borel–Theorem

X compact $\Leftrightarrow$ X closed and bounded

Def X is compact if every open cover of X contains a finite subcover

$\rightarrow$ Compactness is the natural generalization of finiteness

Proposition K_i $i = 1, \ldots, n$ compact sets then $\bigcup_{i=1}^{n} K_i$ is compact.

pf (direct) let $\mathcal{O}$ be an open cover of $\bigcup_{i=1}^{n} K_i$ this $\mathcal{O}$ is also an open cover for each of the K_i's. Since K_i is compact, can find a finite subcover (for K_i) $\mathcal{O}^i_{fin}$ of $\mathcal{O}$.

then $\bigcup_{i=1}^{n} \mathcal{O}^i_{fin}$ will be finite and will cover $\bigcup_{i=1}^{n} K_i$.

$\rightarrow$ Topology is the study of properties using open sets

theorem If $(K \;\;)$ is a sequence of

Theorem If $(K_n)_{n\in\mathbb{N}}$ is a sequence of nested compact sets, $K_n \neq \emptyset \ \forall n$, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$

pf (De Morgan's Laws)

Suppose to the contrary that $\bigcap_{n=1}^{\infty} K_n = \emptyset$

Thus $\{\mathbb{R}\setminus K_n \mid n\in\mathbb{N}\}$ is an open cover for K_1:

(a) $\mathbb{R}\setminus K_n$ is open for all $n\in\mathbb{N}$ ✓

(b) $\bigcup_{n=1}^{\infty}(\mathbb{R}\setminus K_n)$

de Morgan's Laws $\longrightarrow = \mathbb{R}\setminus(\bigcap_{n=1}^{\infty} K_n) = \mathbb{R} \supset K_1$

Since K_1 is compact, $\{\mathbb{R}\setminus K_n \mid n\in\mathbb{N}\}$ has a finite subcover

$$\{\mathbb{R}\setminus K_{n_1}, \mathbb{R}\setminus K_{n_2}, \mathbb{R}\setminus K_{n_3}, \ldots \mathbb{R}\setminus K_{n_j}\}$$

wlog $n_1 < n_2 < n_3 < \ldots n_j$

$$\bigcup_{i=1}^{j} \mathbb{R}\setminus K_{n_i} \supset K_1$$

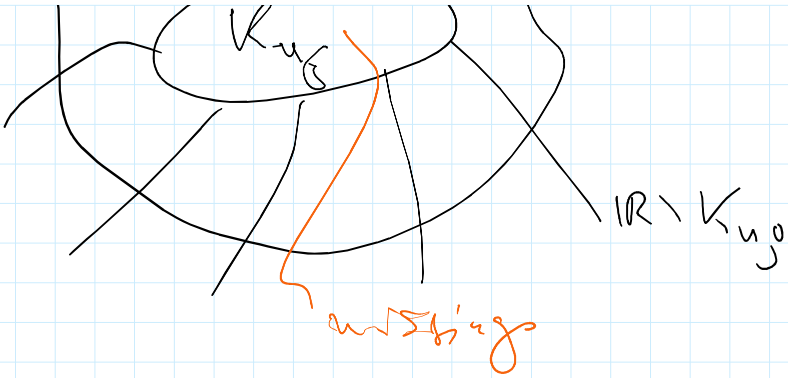
but again

$$\bigcup_{i=1}^{j} \mathbb{R}\setminus K_{n_i} = \mathbb{R}\setminus\bigcap_{i=1}^{j} K_{n_i}$$

$$= \mathbb{R}\setminus K_{n_j} \supset K_1$$

False

reminder: $K_{n_j} \subset K_1$



Def X is <u>sequentially compact</u> if every sequence in X has a convergent subsequence with limit in X.

<u>Heine - Borel Version 2</u>

X compact
$\iff$ X sequentially compact
$\iff$ X is closed and bounded

Pf <u>Step 1</u> X closed and bounded $\Rightarrow$ X is seq'ly compact

Indeed, let (x_n) be any sequence in X. Since X is bounded, (x_n) is bounded and thus, by BW, contains a converging subsequence, say, with limit y.
If $y \notin X$ then y is an acc. pt of X
Since X is closed, $y \in X$.

<u>Step 2</u> X sequ. compact $\Rightarrow$ X is bounded

Suppose X is not bounded. then X contains an unbounded sequence (x_n) such that all of its subsequences are unbounded as well.

So X is not sequentially compact.

Construction of such a sequence (x_n):

Let $n \in \mathbb{N}$. n is not a bound for X so we can find an $x_n \in X$ with

$$|x_n| \geq n.$$

(x_n) is unbounded. For any subsequence (x_{n_k}) we obtain

$$|x_{n_k}| \geq n_k$$

and thus observe that (x_{n_k}) is unbounded.

<u>Step 3</u> X seq'ly compact $\Rightarrow$ X is closed.

If X is not closed, there is an $y \notin X$, y is an acc pt of X then we can find a sequence (x_n) in X converging to $y \notin X$. All of its subsequences converge to $y \notin X$

So X is not sequentially compact.

So X is not sequentially compact.

Connected sets

Connected sets (on the real line) are a topological way to describe intervals

Preliminary Remarks

Def Let A be any set, and let L be the set of it's acc. pts.
Then the closure of A

$$Cl(A) = \overline{A} = A \cup L$$

Thm (HW) If A is any set, $\overline{A}$ is closed.

Examples 1) $A = (0,1)$ $L = [0,1]$

so $\overline{A} = [0,1]$

2) $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$L = \{0\}$

$\overline{A} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

3) $\overline{\mathbb{N}} = \mathbb{N}$

$\mathbb{N}$ $L = \emptyset$

4) $\mathbb{R} \setminus \mathbb{Q}$ $L = \mathbb{R}$

$$\overline{\mathbb{R}\setminus\mathbb{Q}} = \mathbb{R}$$

Def Two sets A and B are called **separated** if $A \neq \emptyset$, $B \neq \emptyset$

$$A \cap \overline{B} = \emptyset \text{ and } \overline{A} \cap B = \emptyset$$

Examples 1) $A = (0,1)$ $B = (1,2)$

separated $\overline{A} \cap B = [0,1] \cap (1,2) = \emptyset$

and $A \cap \overline{B} = (0,1) \cap [1,2] = \emptyset$

2) $A = (0,1)$ $B = [1,2]$

not separated

$\overline{A} \cap B = [0,1] \cap [1,2] = \{1\}$

3) $A = \mathbb{Q}$ $B = \mathbb{R}\setminus\mathbb{Q}$

not separated $\overline{A} \cap B = \mathbb{R}\setminus\mathbb{Q}$

$A \cap \overline{B} = \mathbb{Q}$

Def X is disconnected if it is the union of two separated sets

Def X is connected if it is not disconnected.

Theorem X is connected iff whenever $X = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$

$A \cap B = \emptyset$

and there is a sequence (x_n) in A

and there is a sequence (x_n) in A
converging to an element in B
or there is a sequence (x_n) in B
converging to an element in A.

pf ($\rightsquigarrow$ HW 4)

<u>Theorem</u> X is connected iff
X is an interval.

Def X is an interval if it has the following property:
[If $a, b \in X$ and $a < c < b$
then $c \in X$]

$\Rightarrow$ $a, b \in X$ $\quad a < c < b$

if $c \notin X$

let $A = (-\infty, c) \cap X$
$B = (c, \infty) \cap X$

$A \cup B = X$
$A \cap B = \emptyset$
$A \neq \emptyset$ $\quad B \neq \emptyset$

$\bar{A} \cap B \neq \emptyset$
or $A \cap \bar{B} \neq \emptyset$

but on the other hand

$\bar{A} \cap B \subseteq [-\infty, c] \cap (c, \infty) = \emptyset$
$A \cap \bar{B} \subseteq (-\infty, c) \cap [c, \infty) = \emptyset$
a contradiction

$\Leftarrow$ X is an interval $\Rightarrow$ X is connected

Supp $X = A \cup B$, $A \cap B = \emptyset$

$A \neq \emptyset$, $B \neq \emptyset$

Pick $a_0 \in A$ $b_0 \in B$

$$c_1 = \frac{a_0 + b_0}{2}$$

Case 1 $c_1 \in A$. Set $a_1 = c_1$, $b_1 = b_0$

Case 2 $c_1 \in B$ Set $a_1 = a_0$ $b_1 = c_1$

Continuing in this fashion

$a_n \nearrow \in A$ $b_n \searrow \in B$

$b_n - a_n \to 0$

$a_n \to x \leftarrow b_n$

$x \in A$? $(b_n) \in B$ $b_n \to x$

$x \in B$? $(a_n) \in A$ $a_n \to x$