10/14

Tuesday, October 14, 2025 14:29

Again: X is an interval $\Rightarrow$ X is connected.

pf: Suppose X is an interval, let $A, B \subset \mathbb{R}$

$A \cup B = X$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$.

let $a_0 \in A$, $b_0 \in B$. Wlog $a_0 < b_0$

let $c_1 = \frac{a_0 + b_0}{2} \in X$!

Case 1 $c_1 \in A$ then $a_1 = c_1$, $b_1 = b_0$

Case 2 $c_1 \in B$ then $a_1 = a_0$, $b_1 = c_1$

Continuing

$a_n \nearrow$, $a_n \in A$ $b_n \searrow$, $b_n \in B$

$b_n - a_n \to 0$

thus there is an $x \in X$!

with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$

Case 1 $x \in A$. $b_n \in B$, $b_n \to x \in A$

Case 2 $x \in B$ $a_n \in A$, $a_n \to x \in B$

So X is connected.

Limit Definition

Definition $f : D \to \mathbb{R}$

x_0 is an acc. pt of D

if $\forall \varepsilon > 0$ $\exists \delta > 0$ s. that

if $\forall x \in D$ ~~$0 <$~~ $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$

if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$

$\iff \lim_{x \to x_0} f(x) = L$

Def derivative !

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

can't plug in x_0 !

$$\Delta_{x_0} f = \frac{f(x) - f(x_0)}{x - x_0}$$

$\Delta_{x_0} f : \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$

Note $x_0 \notin \mathbb{R} \setminus \{x_0\}$

But x_0 is an acc. pt. of $\mathbb{R} \setminus \{x_0\}$

Ex 1 $\quad f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ 6 & \text{if } x = 1 \end{cases}$$

$\lim_{x \to 1} f(x) = 2$

Note $\frac{x^2 - 1}{x - 1} = x + 1$ if $x \neq 1$

Pf: Let $\varepsilon > 0$ be given.

We set $\delta = \varepsilon$

Then if $|x - 1| < \delta$, $x \neq 1$

then $|f(x) - 2| = |(x + 1) - 2|$

then $|f(x)-2| = |(x+1)-2|$

$= |x-1| < \delta = \varepsilon$

and done.

Exc 2 $\quad f : \mathbb{R}\setminus\{1\} \longrightarrow \mathbb{R}$

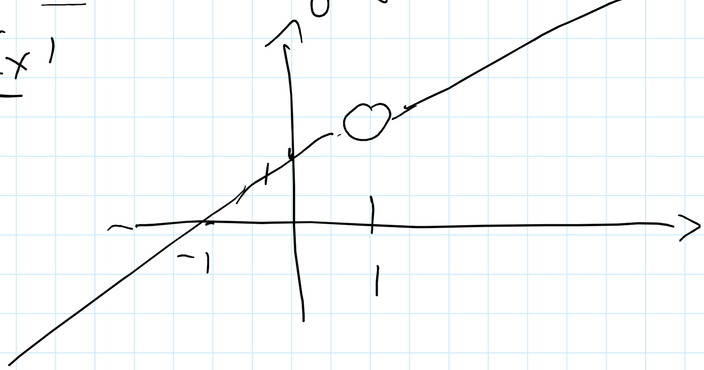
$$f(x) = \frac{x^2-1}{x-1}$$

Note 1 is an acc pt of $\mathbb{R}\setminus\{0\}$

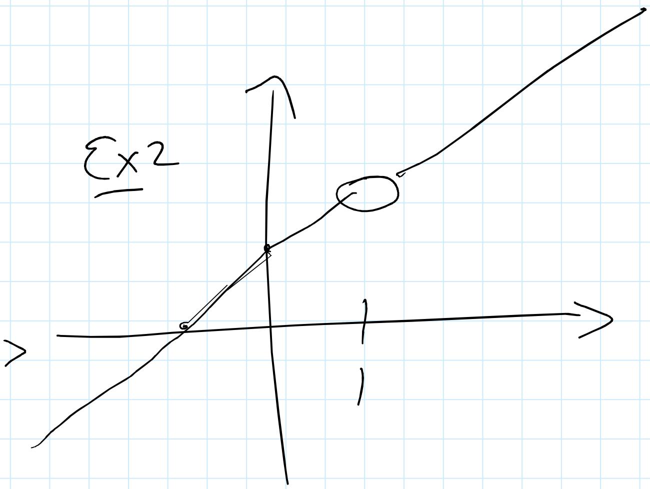
$$\lim_{x\to 1} f(x) = 2$$

Ex1 Ex2 graphs

Ex1



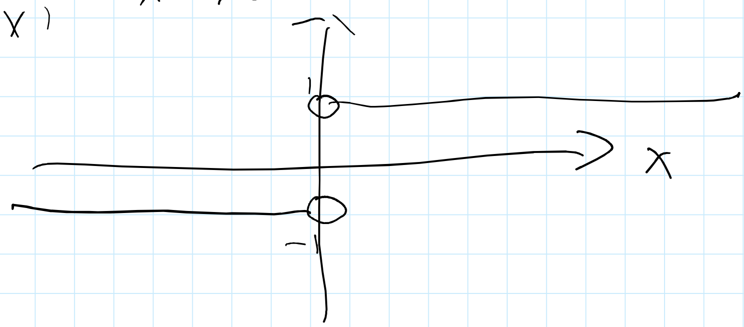
Ex2



Ex3 $\quad f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$

$$f(x) = \frac{|x|}{x} \quad \left(\text{or } f(x) = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}\right.$$

$\lim_{x\to 0} f(x) =$ does not exist



Theorem $\quad f : D \to \mathbb{R}$, x_0 is an acc. pt of D

Then $\lim_{x\to x_0} f(x) = L$ (exists)

Then $\lim_{x \to x_0} f(x) = L$ (exists) (1)

$\iff$ For all sequences $(x_n) \in D$, $x_n \neq x_0$,

$x_n \to x_0$ (2)

we have $f(x_n) \to L$

(converges)

pf (L version)

(1) $\Rightarrow$ (2) Let $\varepsilon > 0$ be given.

Let $(x_n) \in D$, $x_n \neq x_0$, and $x_n \to x_0$

Since $\lim_{x \to x_0} f(x) = L$, there is a $\delta > 0$

s. that if $0 < |x - x_0| < \delta$, $x \in D$,

then $|f(x) - L| < \varepsilon$

Since $x_n \to x_0$, there is an $N \in \mathbb{N}$,

s. that for all $n \geq N$

$|x_n - x_0| < \delta$

Thus for such n,

$|f(x_n) - L| < \varepsilon$

(1) $\Leftarrow$ (2) By contraposition

Assume $\lim_{x \to x_0} f(x) \neq L$

$\left(\neg \left(\forall \varepsilon > 0 \ \exists \delta > 0 : P(x) \Rightarrow Q(x) \right)\right)$

$\iff \left[\exists \varepsilon > 0 \ \forall \delta > 0 : P(x) \wedge \neg Q(x) \right]$

Therefore there is an $\varepsilon > 0$ such that for all

Therefore there is an $\varepsilon > 0$ such that for all

$\delta > 0$ we can find $x : |x - x_0| < \delta$

yet $|f(x) - L| \geq \varepsilon$

Apply this to $\delta = \frac{1}{n}$

we can find $x_n : |x_n - x_0| < \frac{1}{n}$

yet $|f(x_n) - L| \geq \varepsilon$

thus (x_n) - converges to x_0

but $f(x_n)$ does not converge to L.

Going back to Exe 3

$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$$f(x) = \frac{|x|}{x}, \text{ if } x \neq 0$$

$\lim_{x \to 0} f(x)$ does not exist

pick x_n converging to 0

with $x_n < 0$

then $f(x_n) = -1$

so $\lim_{n \to \infty} f(x_n) = -1$

pick y_n converging to 0

with $y_n > 0$

then $f(y_n) = 1 \ \forall n$

so $\lim_{n \to \infty} f(y_n) = 1$

Exa 4 $\quad f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$

$$f(x) = \sin\left(\frac{1}{x}\right)$$

$$\lim_{x \to 0} f(x) = ?$$

even function?

Conjecture: $\lim_{x \to 0} f(x)$ does not exist

when is $f(x) = 1$?

when is $\sin z = 1$?

when $z = \frac{\pi}{2} + 2\pi k \quad (k \in \mathbb{Z})$

$z = \frac{1}{x}$

when $x = \dfrac{1}{\frac{\pi}{2} + 2\pi k}$ then $f(x) = 1$

so pick $x_n = \dfrac{1}{\frac{\pi}{2} + 2n\pi}$, $n \in \mathbb{N}$

for all $n \in \mathbb{N}$: $f(x_n) = 1$

so $x_n \to 0$

$f(x_n) \to 1$

$f(x_n) \to 1$

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when is $\sin z = -1$?

when $z = \frac{3\pi}{2} + 2\pi k$, $(k \in \mathbb{Z})$

$z = \frac{1}{y}$

$f(y) = -1 \iff y = \frac{1}{\frac{3\pi}{2} + 2\pi k}$ $(k \in \mathbb{Z})$

So we pick

$y_n = \frac{1}{\frac{3\pi}{2} - 2\pi n}$ $(n \in \mathbb{Z})$

with $f(y_n) = -1$ for all $n \in \mathbb{N}$

$y_n \to 0$ and $f(y_n) \to -1$