Q: What is Cardinality?

$$A \sim B \iff \exists \varphi : A \to B \text{ bijection}$$

Q: De Morgan's Laws?

$$\mathbb{R} \setminus \bigcup_{\alpha \in A} B_\alpha = \bigcap_{\alpha \in A} (\mathbb{R} \setminus B_\alpha)$$

Q: ~~Heine~~ - Borel!

A set K is compact $\iff$ K is closed and bounded

($K \subseteq \mathbb{R}$)

Pf. 1) K compact $\Rightarrow$ K closed

2) K compact $\Rightarrow$ K bounded

3) K closed and bounded $\Rightarrow$ K compact

3) for $K = [0, 1]$

$\hookrightarrow$ contradiction argument

If K is not compact, there is an

(*) open cover for K which does not admit a finite subcover.

$a_0 = 0$ $\qquad$ $b_0 = 1$

$$c_1 = \frac{a_0 + b_0}{2} = \frac{1}{2}$$

For $[a_0, c_1]$ or $[c_1, b_0]$

the open cover will not have a finite subcover for that smaller interval

Can pass to $[a_1, b_1]$, $b_1 - a_1 = \frac{1}{2}$

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with (*)

Continuing this procedure

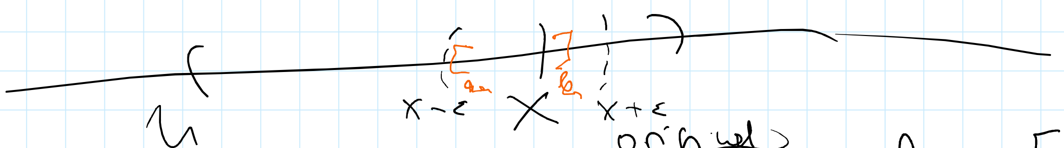
we get

$a_n \nearrow$, $b_n \searrow$, $b_n - a_n \to 0$

and $[a_n, b_n]$ satisfies (*) $\forall n$

By MCT, there is an $x \in [0,1]$

such that $a_n \to x$, $b_n \to x$



Pick U in the ~~open~~ original cover for $[0,1]$

Can find $\varepsilon > 0$ s. that

$(x - \varepsilon, x + \varepsilon) \subset U$

(remember U is open)

Since $b_n - a_n \to 0$

eventually $[a_n, b_n] \subset (x - \varepsilon, x + \varepsilon)$

$\subset U$

a contradiction!

HBT : K closed and bounded

$\iff$ K sequentially compact

Def K is sequentially compact

if every sequence in K

has a converging subsequence

with limit in K.

Q Exercise 3.4.5

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$A, B \neq \emptyset$ If $\exists$ open sets U and V $A \subseteq U$ $B \subseteq V$ $U \cap V = \emptyset$ then A and B are separated

i.e. $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$

Note: ① $B \subseteq U^c$, U^c closed

and v.v. ② $A \subseteq V^c$, V^c closed

① $B \cap \overline{A} = \emptyset$ $\overline{A} \subset U^c$

② ...

need: $C \subseteq D \Rightarrow \overline{C} \subseteq \overline{D}$

Q: Find all acc pts of

$M = \{ \frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N} \}$

answer: 0, $\frac{1}{n}$ $(n \in \mathbb{N})$

ⓐ 0 is an acc pt of M

$(\frac{1}{n}) \in M$, $\frac{1}{n} \to 0$ ✓

ⓑ $\frac{1}{n}$ is an acc pt of M for any $n \in \mathbb{N}$

$(\frac{1}{n} + \frac{1}{m})_{m \in \mathbb{N}} \in M$

$\frac{1}{n} + \frac{1}{m} \longrightarrow \frac{1}{n}$ ✓

(c) Let $\left(\frac{1}{n_k} + \frac{1}{m_k}\right) \in M$ of distinct elements

Note all these sequences converge

Rewrite!

If (n_k) have a constant subseq n_{k_ℓ}
& (m_k) have a constant subseq m_{k_ℓ}
then $\left(\frac{1}{n_{k_\ell}} + \frac{1}{m_{k_\ell}}\right)$ is eventually constant, contradicting that the sequence has all distinct elements

m_k's eventually constant

If $n_k \to \infty$
then $\lim_{k\to\infty} \frac{1}{n_k} + \frac{1}{m_k} = \lim_{k\to\infty} \frac{1}{m_k} = \begin{cases} \frac{1}{m_k} \\ 0 \end{cases}$

$m_k \to 0$

If $m_k \to \infty$
then $\lim_{k\to\infty} \frac{1}{n_k} + \frac{1}{m_k}$
$= \lim_{k\to\infty} \frac{1}{n_k} = \begin{cases} 0 & \text{if } n_k \to \infty \\ \frac{1}{n_{k^*}} & \text{if } (n_k) \text{ becomes eventually constant} \end{cases}$

Q The Cantor set is closed

$C_0 = [0, 1]$ closed

$C_1 = C_{10} \cup C_{11}$ where $C_{10} = [0, \frac{1}{3}]$
$C_{11} = [\frac{2}{3}, 1]$

$C_2 = C_{20} \cup C_{21} \cup C_{22} \cup C_{23}$

$\vdots$

Cantor set $C = \bigcap_{k=1}^{\infty} C_k$

C_1 is closed as the <u>finite</u> union of 2 closed sets

C_k is closed as the finite union of closed sets

C is closed as the intersection of closed sets

<u>need</u> 1) C and D are closed $\Rightarrow C \cup D$ is closed

2) C_α, ~~$\alpha \in A$, are~~ closed sets then $\bigcap_{\alpha \in A} C_\alpha$ is closed

Q why C?

It is an interesting set

→ closed, bounded and thus compact

→ does not contain an open interval

→ uncountable : $\mathbb{R} \sim C$

→ C is perfect

→ length C is zero.

<u>Q:</u> A closed, not compact?

→ $\mathbb{N}$

A bounded, not compact?

→ $(0, 1)$

$\longrightarrow (0,1)$

Q Is the set $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ compact?

set is not closed, therefore not compact

open cover without finite subcover?

$U_0 = (\frac{1}{3}, 2)$, $U_n = (0, \frac{1}{n^2})$, $n \in \mathbb{N}$

review the pf

not closed $\Rightarrow$ not compact

sequence with no conv. subsequence whose limit ~~outside~~ inside the ~~set~~?

ex. $\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5} \dots \longrightarrow 0$

Q K compact F is closed

(a) $K \cap F$ is closed, is compact as the closed subset of K

(b) $\overline{F^c \cup K^c}$ closed, $F^c \cup K^c$ is open, ~~not~~ bounded, so

(c) $K \setminus F$ K, F closed, thus $K \setminus F$ open not cpt.

(d) $\overline{K \cap F^c}$ is closed and bounded ✓

Need: If A is bounded, $\overline{A}$ is bounded

Pf A is bounded

$\Rightarrow A \subseteq [a,b]$ for some $a, b \in \mathbb{R}$

so $\overline{A} \subseteq \overline{[a,b]} = [a,b]$ ✓

Q $(a,b) \sim \mathbb{R}$

$(-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$

$f : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$

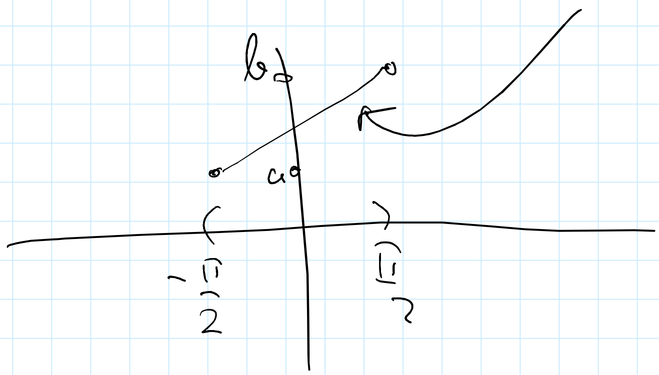
$$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

$f(x) = \tan x$ is a bijection.

Let's find ~~a~~ linear function

$$g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow (a, b)$$

onto



Q Difference b/w open and closed sets

open sets ⟶ open intervals

closed sets ⟵ closed interval

$\emptyset, \mathbb{R}$ are the only sets that are both open and closed

Most sets are neither,

ex. $[0, \frac{1}{2})$

Complementary:

A closed $\Rightarrow \mathbb{R} \setminus A$ open

and vice versa

Q a) $s \in \overline{A}$ $\quad s = \sup A$

on the first test

b) Can an open set contain its supremum?

A has sup $\Rightarrow A \neq \emptyset$

$$(\underset{s \to}{\cdot}) \subset A$$