10/28

Tuesday, October 28, 2025 13:48

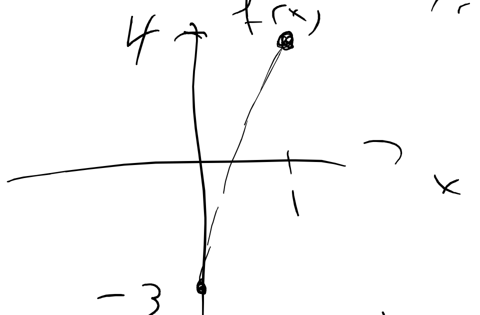
**P4** Let $(a_n)$ be bounded sequence s.that every conv. subsequence converges to the same limit. Then $(a_n)$ itself converges to that limit.
(HW problem)

Pf ($(a_n)$ is bounded is necessary!)
By BW $(a_n)$ has a convergent subsequence, say with limit $L$.
Suppose to the contrary that $(a_n)$ does not converge to $L$.
Then there is an $\varepsilon > 0$ and a subsequence $(a_{n_k})$ of $(a_n)$ such that

$$|a_{n_k} - L| \geq \varepsilon \quad \forall k \in \mathbb{N}$$

$(a_{n_k})$ is bounded, so by BW it will a subsequence that converges, albeit not to $L$. Note this subsequence of $(a_{n_k})$ is also a subsequence of $(a_n)$.

**P3.2** $(0,1)$ and $(-3,4)$ have the same cardinality:




$f(x)$ is the line passing thru pts $(0,-3)$ and $(1,4)$

**P3.3**

$$\left.\begin{array}{l} A \sim \mathbb{N} \\ B \sim \mathbb{N} \end{array}\right\} \overset{!}{\Longrightarrow} A \cup B \sim \mathbb{N}$$

$B - \mathbb{N}$

<u>Case 1</u> $A \cap B = \emptyset$

$A = \{a_1, a_2, a_3, \ldots\}$

$B = \{b_1, b_2, b_3, \ldots\}$

$f: \mathbb{N} \to A$ is a bijection

$a_1 = f(1)$

$a_2 = f(2)$

$\vdots$

$A \cup B = \{a_1, b_1, a_2, b_2, \ldots\}$

$g: \mathbb{N} \to A \cup B$

$\left.\begin{array}{l} g(2k-1) = a_k \\ g(2k) = b_k \end{array}\right\}$ bijection

<u>Case 2</u> $A \cap B \neq \emptyset$

<u>Case 2A</u> $B \setminus A$ is infinite

↳ omit the doubletons

$A \cup B = \{a_1, b_1, a_2, \cancel{\phantom{b}}, a_3, b_3$

↑ if $b_2 \in B \setminus A$

<u>Case 2B</u> $B \setminus A$ is finite

$A \cup B = \{\underbrace{b_1, \ldots, b_n}_{\in B \setminus A}, a_1, a_2, a_3, \ldots\}$

<u>PS. 2</u>

$A = \{1, 2, 3\}$ finite

$B = [0, 2] \setminus \{1\}$ ⟵

$B = [0,2] \setminus \{1\}$ ⟵

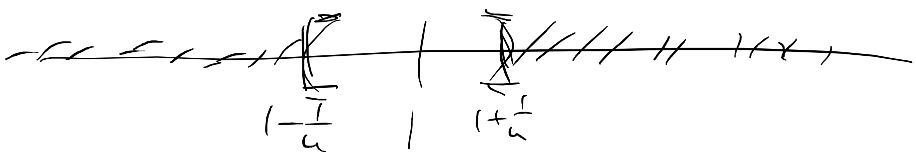
$C = [0,3] \setminus (1,2)$

$= [0,1] \cup [2,3]$ (compact (closed, bounded))

$B$ is not compact

1 is an acc. pt. of $B$,

but $1 \notin B$




$U_n = \mathbb{R} \setminus [1-\frac{1}{n}, 1+\frac{1}{n}]$ open

form an open cover

$$\bigcup_{n=1}^{\infty} U_n = \bigcup_{n \in \mathbb{N}} \mathbb{R} \setminus [1-\frac{1}{n}, 1+\frac{1}{n}]$$

$$\overset{\text{d'M}}{=} \mathbb{R} \setminus \bigcap_{n \in \mathbb{N}} [1-\frac{1}{n}, 1+\frac{1}{n}]$$

$$= \mathbb{R} \setminus \{1\} \supseteq B$$

what about finite subcollection of $(U_n)_{n \in \mathbb{N}}$

say $\{U_{n_1}, \dots U_{n_k}\}$

$n_1 < n_2 < \dots n_k$

$$\bigcup_{i=1}^{k} U_{n_i} = U_{n_k} = \mathbb{R} \setminus [1-\frac{1}{n_k}, 1+\frac{1}{n_k}]$$

this is not a cover.

contains elements of $B$, b/c 1 is an acc pt of $B$

P6 was also a HW problem!

Last time

Let $f: D \to \mathbb{R}$ be a function, $x_0$ acc. pt of $D$

We say $\lim_{x \to x_0} f(x) = L$

if $\forall \varepsilon > 0$ $\exists \delta > 0$ s. that

if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$

$x \in D$

Definition of Derivative

$f: \mathbb{R} \to \mathbb{R}$

$\Delta: \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$

$$\Delta(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\lim_{x \to x_0} \Delta(x) = f'(x_0)$$

Ex: $f: [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Q $x_0 \in [0,1]$ $\lim_{x \to x_0} f(x) = ?$ No limit

Thm $f: D \to \mathbb{R}$, $x_0$ a acc pt of $D$

T.f.a.e.

(1) $\lim_{x \to x_0} f(x) = L$

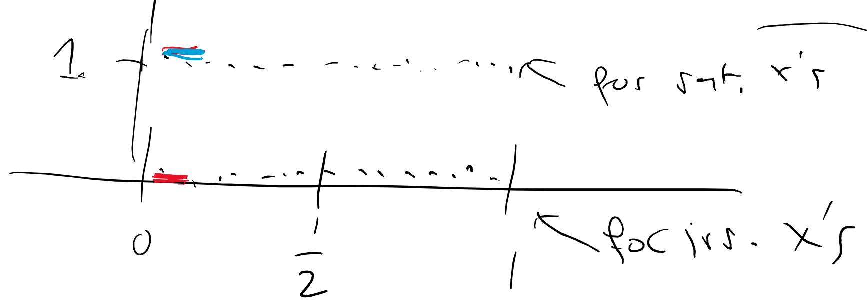
(2) Whenever $(x_n)$ in $D$, $x_n \neq x_0$ $\forall n \in \mathbb{N}$

$x_n \to x_0$

[illegible]

$x_n \longrightarrow x_0$

then $f(x_n) \longrightarrow L$




(cp. $a_n = (-1)^n$ )

let $x_n \in [0,1] \cap \mathbb{Q}$ with $x_n \longrightarrow x_0$

then $f(x_n) = 1$ for all $n \in \mathbb{N}$

so $f(x_n) \longrightarrow 1$

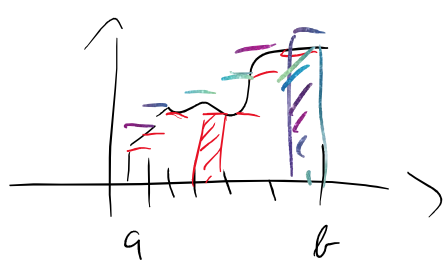
let $y_n \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$ with $y_n \longrightarrow x_0$

then $f(y_n) = 0$ for all $n \in \mathbb{N}$

so $f(y_n) \longrightarrow 0$

therefore $\lim_{x \to x_0} f(x)$ does not exist

Integration of $f(x)$




$f$ is Riemann-integrable if

$\lim L\text{-}\int f(x)\,dx$

$= \lim U\text{-}\int f(x)\,dx$

For our $f$ upstairs (Dirichlet function)

$L\text{-}\int f(x)\,dx = 0$

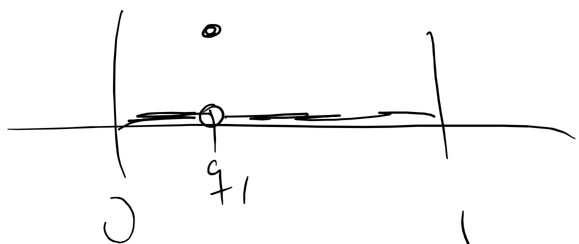
$$L-\int_0^1 f(x)\,dx = 0$$

$$U-\int_0^1 f(x)\,dx = 1$$

Thus $f$ is not Riemann-integrable.

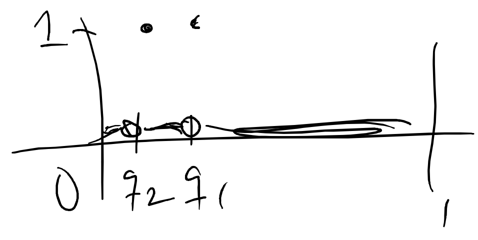
$$\mathbb{Q}\cap[0,1] = \{q_1, q_2, q_3, \dots\}$$

$$f_1(x) = \begin{cases} 1 & \text{if } x = q_1 \\ 0 & \text{if } x \neq q_1 \end{cases}$$




$$\int_0^1 f_1(x)\,dx = 0$$

$$f_2(x) = \begin{cases} 1 & \text{if } x = q_1 \text{ or } x = q_2 \\ 0 & \text{elsewhere} \end{cases}$$




$$\int_0^1 f_2(x)\,dx = 0$$

Fix $x \in [0,1]$ $\lim_{n\to\infty} f_n(x) = g(x)$

where $g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$

= Dirichlet function

$f_n \nearrow f$ (ptw.)

$f_n \xrightarrow{(\text{ptw.})} f$

$\int f_n = 0$ $\quad$ $\int f$ does not exist

very undesirable !!

How to fix that ??

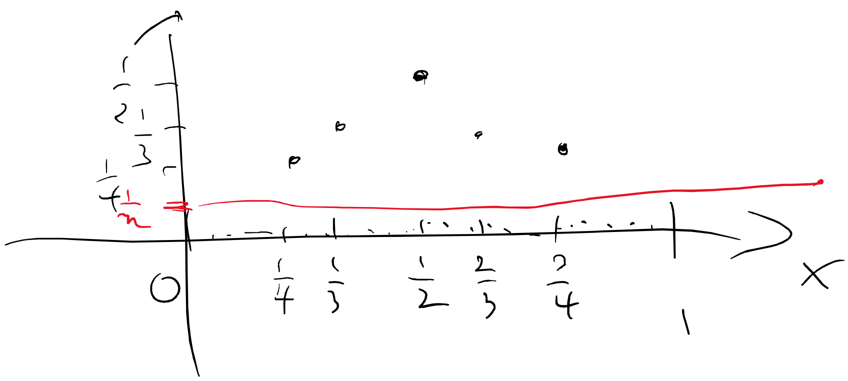
~ 70 years later

Lebesgue - integral (1907)

<u>Exc 2</u> Thomae function

$f : (0,1) \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q},\ p,q \in \mathbb{Z},\ \gcd(p,q) = 1 \end{cases}$$




<u>Q</u> Fix $x_0 \in [0,1]$

$$\lim_{x \to x_0} f(x) = 0$$

<u>Claim</u> let $F_n = \{ x \in (0,1) \mid f(x) \geq \frac{1}{n} \}$ is finite!

$$F_n = \{ \tfrac{1}{2};\ \tfrac{1}{3}, \tfrac{2}{3};\ \tfrac{1}{4}, \tfrac{3}{4};\ \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5};\ \dots\ \tfrac{1}{n}\ \tfrac{2}{n}\ \dots\ \tfrac{n-1}{n} \}$$

$\ldots \frac{1}{n}, \frac{2}{n} \ldots \frac{n-1}{n} \}$

let $G_n = F_n \setminus \{x_0\}$

$G_n = \{g_1, g_2, \ldots g_k\}$

let $x_0$ be given,

let $\delta_i = |x_0 - g_i| > 0$

and let $\delta = \min_{i=1,\ldots,k} \delta_i > 0$

Now if $|x - x_0| < \delta$

then $x \notin G_n$

$\Rightarrow f(x_n) < \frac{1}{n}$

i.e. $|f(x_n) - 0| < \frac{1}{n}$

showing that $\lim_{x \to x_0} f(x) = 0$