About P21, 22 on the homework:

given 2 Cauchy sequences of rational numbers

we say $(a_n) \sim (b_n)$ "equivalent"

if $\lim_{n\to\infty} |a_n - b_n| = 0$

(Def (a_n) Cauchy sequence if $\forall k \in \mathbb{N}\ \exists N \in \mathbb{N}$ s.that $\forall n, m \geq N : |a_n - a_m| < \frac{1}{k}$)

Examples $(a_n) = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ $(b_n) = \left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$

$(a_n) \sim (b_n)$

Exercise This defines an equivalence relation!

$(a_n)_\sim = \{ (b_n)$ Cauchy sequences of rat. #s $(a_n) \sim (b_n) \}$.

Def a real number is an equivalence class of a Cauchy sequence of rational numbers

Ex $\overset{a_1}{1}, \overset{a_2}{1.4}, \overset{a_3}{1.41}, \overset{a_4}{\dots}$

Cauchy sequence

$\sqrt{2}$ to be $(a_n)_\sim$

" Completion of the rational numbers "

~ 1860s by Cauchy, Dedekind

Def (Thomae function)

$$f : (0,1) \to \mathbb{R}$$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q},\ p, q \in \mathbb{Z},\ \gcd(p,q) = 1 \end{cases}$$

$\lim_{x \to x_0} f(x) = 0$ for all $x \in [0,1]$

Caveat $f : \{1\} \to \mathbb{R}$

$f(1) = 5$

$\lim_{x \to 1} f(x)$ does not exist!

Exa $f : [0, \infty) \to \mathbb{R}$

$f(x) = \sqrt{x}$

Case 1 $x_0 = 0$ $\lim_{x \to 0} \sqrt{x} = 0$

Aside: $|f(x) - f(0)| = \sqrt{x} < \varepsilon \Longleftarrow x < \varepsilon^2 = \delta$

Pf given $\varepsilon > 0$. Choose $\delta = \varepsilon^2 > 0$

then if $0 < x < \delta$ then $f(x) < \varepsilon$ as required.

Case 2 $x_0 \neq 0$

$$|f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x_0}|$$

$$= \frac{|\sqrt{x} - \sqrt{x_0}|(\sqrt{x} + \sqrt{x_0})}{\sqrt{x} + \sqrt{x_0}} = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}}$$

$$< \frac{|x - x_0|}{\sqrt{}} < \varepsilon$$

$$< \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon$$

$$\Leftarrow \quad |x - x_0| < \varepsilon\sqrt{x_0} = \delta$$

Pf Let $\varepsilon > 0$ be given. Set $\delta = \varepsilon\sqrt{x_0} > 0$

then if $|x - x_0| < \delta$

$$\iff |x - x_0| < \varepsilon\sqrt{x_0}$$

$$\iff \frac{|x - x_0|}{x_0} < \varepsilon$$

$$\vdots$$

$$|\sqrt{x} - \sqrt{x_0}| < \varepsilon$$

<u>Algebra of limits</u>

Suppose $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$

x_0 acc. pt of D

$$\lim_{x \to x_0} f(x) = L$$

$$\lim_{x \to x_0} g(x) = M$$

then

1) $\lim_{x \to x_0} (f(x) + g(x)) = L + M$

2) $\lim_{x \to x_0} (f(x) \cdot g(x)) = L \cdot M$

3) If $g(x) \neq 0 \ \forall x \in D$ & if $M \neq 0$

then

$$\lim_{x \to x_0} \frac{1}{g(x)} = \frac{1}{M}$$

4) If $f(x) \leq g(x) \ \forall x \in D$

then $L \leq M$.

Pf Use sequence characterization of existence then ... (cut off)

Pf use sequence characterization of existence of a limit and the corresponding results about sequences

Continuity

Let $f : D \to \mathbb{R}$ be a function, $x_0 \in D$

then f is continuous at x_0 if for all $\varepsilon > 0$ there is a $\delta > 0$ s. that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ $(x \in D)$

replaces: x_0 is an acc pt of D

no more needed that $x \neq x_0$

instead of L

Remark $f : D \to \mathbb{R}$

$x_0 \in D$ but not an acc. pt of D

then f is continuous at x_0 !

why? If x_0 is not an acc pt of D there is a $\delta > 0$ such that

$$V_\delta(x_0) \cap D = \{x_0\}$$

[i.e. for this δ, x_0 is the only x one has to check in the definition.

On the other hand

if x_0 is an acc. pt of D (and in D)

then $f(x)$ is continuous at x ...

if x_0 is an ... of ...

then $f(x)$ is continuous at x_0

$\iff \lim_{x \to x_0} f(x) = f(x_0)$

Sequ characterization

$f : D \to \mathbb{R}$, $x_0 \in D$

f is continuous at x_0

$\iff \forall (x_n) \in D$ with $x_n \to x_0$,

the sequence $f(x_n) \to f(x_0)$

Algebra of continuous functions

$f : D \to \mathbb{R}$ $\quad g : D \to \mathbb{R}$

$x_0 \in D$ $\quad f, g$ cont at x_0

1) $f + g$ cont. at x_0

2) $f \cdot g$ cont. at x_0

3) If $g(x) \neq 0$ for all $x \in D$

$\frac{1}{g}$ cont. at x_0

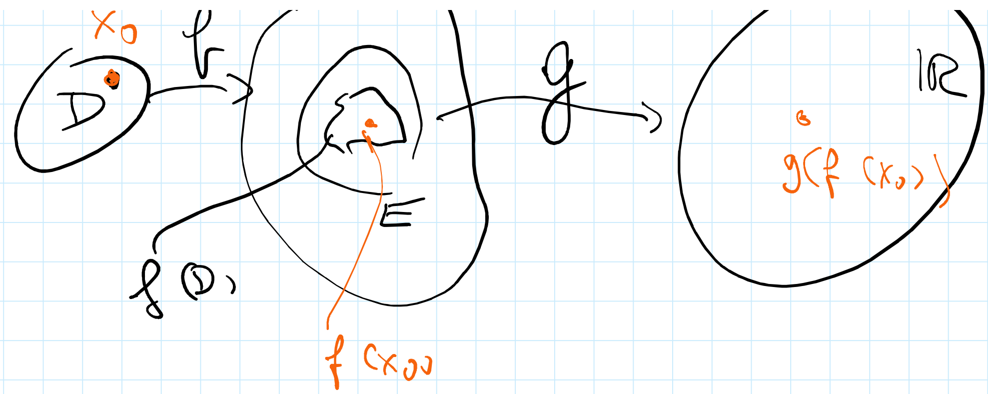
4) composition of cont. fcts. is cont.

4) let $f : D \to \mathbb{R}$ continuous at $x_0 \in D$

let $g : E \to \mathbb{R}$ cont. at $y_0 = f(x_0) \in E$

$f(D) \subset E$

then $g \circ f : D \to \mathbb{R}$ is continuous at x_0



1st pf: let (x_n) be a sequence in D

$$x_n \longrightarrow x_0$$

Since f is continuous at x_0,

$$\underset{\in E}{f(x_n)} \longrightarrow \underset{\in E}{f(x_0)}$$

Since g is continuous at $y_0 = f(x_0)$

$$g(f(x_n)) \longrightarrow g(f(x_0)).$$

2nd pf (ε-δ machinery)

Let $\varepsilon > 0$ be given. Since g is continuous at y_0, there is a $\delta' > 0$

s. that if $|y - y_0| < \delta'$
$(y \in E)$
then $|g(y) - g(y_0)| < \varepsilon$ $\quad$ (*)

Since f is continuous at $x_0 \in D$

there is a $\delta > 0$ s. that

$$\Big(\underset{x \in D}{|x - x_0| < \delta} \implies |f(x) - f(x_0)| < \delta'$$

So if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta'$

So if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta$

then by ($\cancel{*}$) $|g(f(x)) - g(f(x_0))| < \varepsilon$

$$\left(g \circ f \quad \begin{matrix} \xleftarrow{\quad} \text{sequence} \\ \xrightarrow{\quad} \epsilon - \delta \end{matrix} \right)$$

Exa 1) Polynomials are continuous everywhere

2) Thomae function

$$f : (0,1) \to \mathbb{R}$$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irr.} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

$$\lim_{x \to x_0} f(x) = 0 \quad \forall x \in [0,1]$$

Q? Is f continuous at $x_0 = \frac{1}{2}$?

NO $\lim_{x \to \frac{1}{2}} f(x) = 0$ but $f(\frac{1}{2}) = \frac{1}{2}$

in general

f is continuous on $\mathbb{R} \setminus \mathbb{Q}$

discontinuous on $\mathbb{Q}$

Next question

Is there a function

continuous on $\mathbb{Q}$

discontinuous on $\mathbb{R} \setminus \mathbb{Q}$?

discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.

NO !????

$\hookrightarrow$ R. L. Baire ~ 1900

Def If $f : D \to \mathbb{R}$ is continuous for all $x \in D$

we will say "f is continuous on D"

Theorem If $f : K \to \mathbb{R}$ is continuous on K and K is compact,

then $f(K)$ is compact.

$f(K) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in K\}$

"image of K under f"

pf Let (y_n) be a sequence in $f(K)$

We have to show that (y_n) has a converging subsequence (y_{n_k}) with limit in K.

So, let (y_n) be a sequence in $f(K)$

Let $x_n \in K$ be chosen such that

$f(x_n) = y_n$

Since K is compact, (x_n) has a converging subsequence (x_{n_k}) with limit $x_0 \in K$

Since f is continuous on K,

Since f is continuous on K,

$(f(x_{n_k}))$ will converge to $f(x_0)$

$\overset{\shortparallel}{(y_{n_k})} \longrightarrow \underline{y_0 = f(x_0)}$

$y_0 \in f(K)$.

Corollary **MinMax Theorem**

A continuous function on a compact set attains its minimum and its maximum.

Pf $f : K \to \mathbb{R}$ K compact
f continuous on K.

By theorem, $f(K)$ is compact
i.e. closed and bounded.

Since $f(K)$ is bounded, $f(K)$ has a sup, call it α. Since $f(K)$ is closed, $\alpha \in f(K)$.

there is an $x_1 \in K$
s. that $f(x_1) = \alpha = \sup f(K)$

i.e. $f(x_1) \geq f(x)$
for all $x \in K$.

Similarly there is an $x_0 \in K$
s. that $f(x_0) \leq f(x)$
$\forall x \in K$

s.that $f(x_0) \leq f(x)$
$\forall x \in K$.