Def $f: A \to \mathbb{R}$ is uniformly continuous on A
if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.that $\forall x, y \in A$
$|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$

(1) f unif'ly cont. on A: $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in A$
$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

(2) f cont. on A: $\forall y \in A \ \forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A$
$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

In (2) δ may depend on the choice of y
in (1) we have to decide on the δ before we know y.
(δ is independent of the y)

thm uniform continuity $\Rightarrow$ continuity

Q? cont $\Rightarrow$ uniform continuity?

what does it mean f is not unif'ly cont. on A?
$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x, y \in A$
$|x-y| < \delta$ yet $|f(x) - f(y)| \geq \varepsilon$

$\Leftrightarrow \exists \varepsilon > 0 \ \exists x_n, y_n \in A$
$|x_n - y_n| < \frac{1}{n}$ yet $|f(x_n) - f(y_n)| \geq \varepsilon$

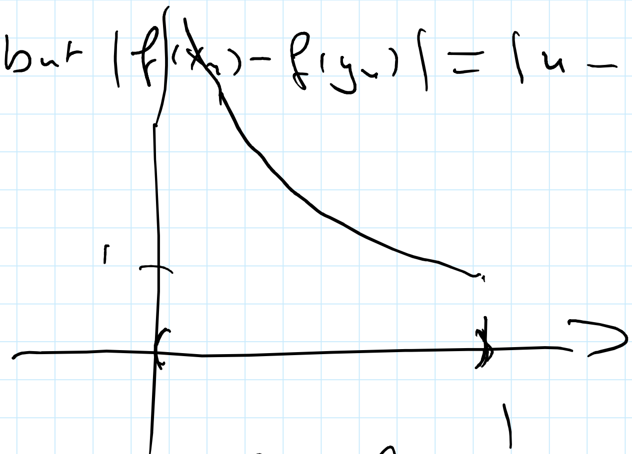
Exa $f: (0,1) \to \mathbb{R}$ is cont. on $(0,1)$
$f(x) = \frac{1}{x}$ is not unif'ly cont. on $(0,1)$

Pf $\varepsilon = 1$ $x_n = \frac{1}{n}$ $y_n = \frac{1}{n+1}$

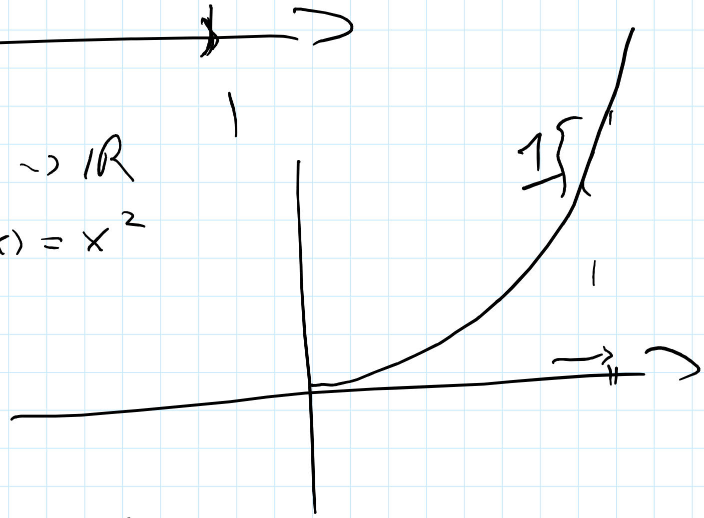
$|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n}$

$|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n}$

but $|f(x_n) - f(y_n)| = |n - (n+1)| = \underline{1}$



Ex: $f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$



<u>Theorem</u> $f : K \to \mathbb{R}$ If f is cont. on K and if K is <u>compact</u> then f is unif'ly cont. on K.

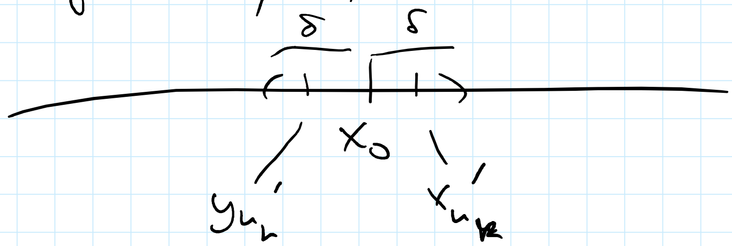
<u>Pf</u> Suppose to the contrary that f is <u>not</u> unif'ly cont. on K.

So, there is an $\varepsilon > 0$, and $x_n, y_n \in A$
$|x_n - y_n| < \frac{1}{n}$ yet $|f(x_n) - f(y_n)| \geq \varepsilon$

Since K is compact, (x_n) has a subsequence (x_{n_k}) converging to some element x_0 in K. (y_{n_k}) also has a converging subsequence (y'_{n_k}). <u>Note</u>: (y'_{n_k}) also converges to x_0.

given $\varepsilon > 0$, for all $\delta > 0$ I can find x'_{n_k}, y'_{n_k}

given $\varepsilon > 0$, for all $\delta > 0$ I can find x_{n_k}', y_{n_k}'



$$|x_{n_k}' - x_0| < \delta$$

and $$|y_{n_k}' - x_0| < \delta$$

but $$|f(x_{n_k}') - f(y_{n_k}')| \geq \varepsilon$$

this implies $$|f(x_{n_k}') - f(x_0)| \geq \varepsilon/2$$

$$|f(y_{n_k}') - f(x_0)| \geq \frac{\varepsilon}{2}$$

this implies that f is <u>not</u> continuous at x_0.

<u>Remark</u> $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ is not unif'ly cont. on $\mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ $f(x) = x$ <u>is</u> unif'ly cont. on $\mathbb{R}$

(choose $\delta = \varepsilon$)

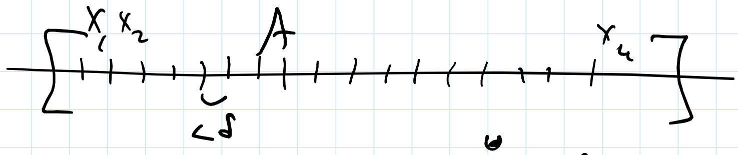
So the product of 2 unif'ly cont. fcts is not necessarily unif'ly continuous.

<u>Theorem</u> $f : A \to \mathbb{R}$, f unif'ly cont. on A. If A is bounded then $f(A)$ is Bounded

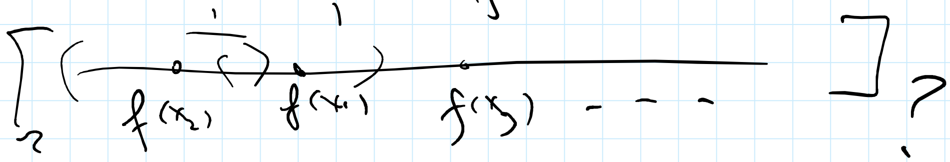
<u>Pf</u> Let $\varepsilon = \underline{1}$ be given. There is a $\delta > 0$ s.that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < 1$

Since A is bounded, we can find finitely many $x_1, \ldots, x_n \in A$

finitely many $x_1, \dots, x_n \in A$
s. that $\{V_\delta(x_i) \mid i=1,\dots,n\}$ is
an open cover



what does this imply for $f(A)$?
$\{V_1(f(x_i)) \mid i=1,\dots,n\}$ will cover $f(A)$
and thus $f(A)$ is bounded



Definition: $f: A \to \mathbb{R}$ is called a Lipschitz function if there is an $M > 0$ s. that $\forall x,y \in A$

$$|f(x) - f(y)| \leq M \cdot |x-y|$$

f diff. with bounded derivatives $\Rightarrow$ Lipschitz $\overset{(1)}{\Rightarrow}$ unif'ly cont $\Rightarrow$ cont

false (A cpt)

Pf (1) Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{M}$

then if $|x-y| < \frac{\varepsilon}{M}$,

$$|f(x) - f(y)| \leq M \cdot |x-y| < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

next HW problem:

$f: [0,1] \to \mathbb{R}$

$f(x) = \sqrt{x}$

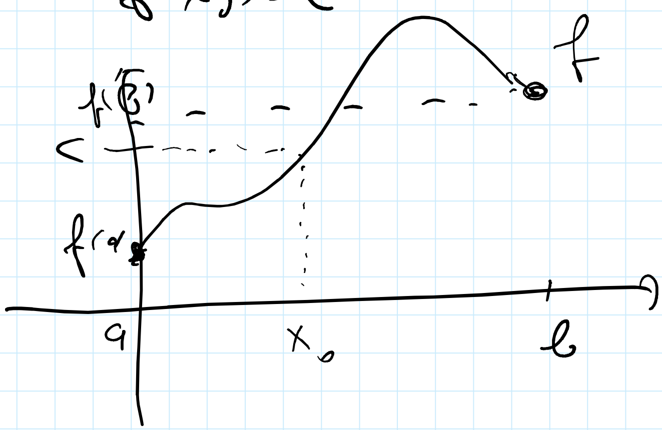
then f is unif'ly cont., but not Lipschitz then f is unif. cont., but not Lipschitz

Intermediate Value Theorem

$f : [a,b] \to \mathbb{R}$ cont. on $[a,b]$ and let c be b/w $f(a)$ and $f(b)$

Then there is an $x_0 \in (a,b)$ s.th.

$$f(x_0) = c$$



proof: $f(a) < c$ $\quad$ $f(b) > c$ $\quad$ wlog.

$a_0 = a$ $\quad$ $b_0 = b$

$$m_1 = \frac{a_0 + b_0}{2}$$

If $f(m_1) = c$ we're done

If $f(m_1) < c$, we set $a_1 = m_1$, $b_1 = b_0$

If $f(m_1) > c$, we set $a_1 = a_0$, $b_1 = m_1$

Continuing in this fashion – unless we stop –

we get $(a_n) \nearrow$ $\quad$ $(b_n) \searrow$

$$b_n - a_n \to 0$$

Thus (a_n) and (b_n) converge to

thus (a_n) and (b_n) converge to a common limit x_0.

Claim $f(x_0) = c$

$a_n \to x_0$

$f(a_n) \to f(x_0)$ by continuity

$f(a_n) \leq c \Rightarrow f(x_0) \leq c$

$b_n \to x_0$

$f(b_n) \to f(x_0)$

$f(b_n) \geq c \Rightarrow f(x_0) \geq c$

$\Big\}\ f(x_0) = c$

Ex: $f(x) = x^2$ cont. everywhere

$f(1) = 1 \quad f(2) = 4$

$c = 2$

IVT delivers $x_0 = \sqrt{2}$

"bisection method"

very slow

fastest method

Newton-Raphson Method

Theorem $f: A \to \mathbb{R}$ continuous on A

If A is connected then $f(A)$ is also connected.

Remark $A = [a, b]$

then we know $f(A)$ is connected

$f(A)$ is compact

i.e. $f(A)$ is closed

i.e. $f(A)$ is closed and bounded

$\Longrightarrow f(A) = [c, d]$ for some c and d

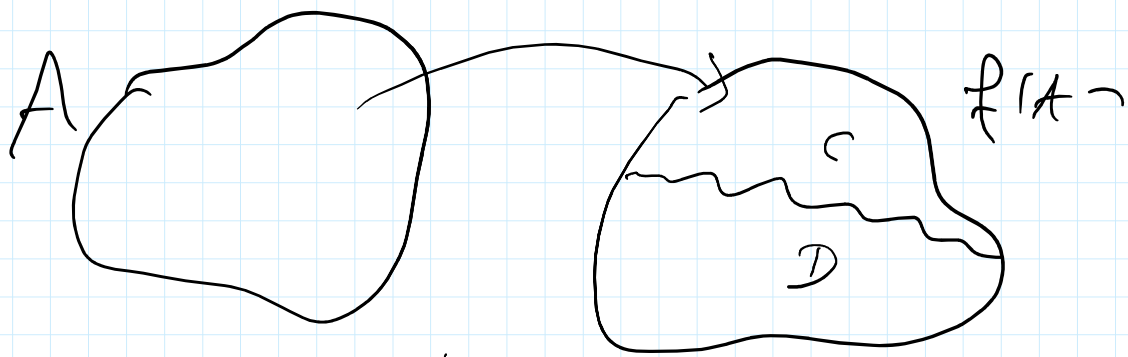
Note that this implies the IVT!

Pf We want to show that $f(A)$ is connected.

Consider C, D s.t. $C \cup D = f(A)$

$C \neq \emptyset$, $D \neq \emptyset$, $C \cap D = \emptyset$

we will find a sequence (y_n) in C with limit in D or vice versa.



Consider $f^{-1}(C) \subset A$

$f^{-1}(D) \subset A$

$f^{-1}(C) = \{x \in A \mid f(x) \in C\}$

Note $f^{-1}(C) \cup f^{-1}(D) = A$

$f^{-1}(C) \cap f^{-1}(D) = \emptyset$

$f^{-1}(C) \neq \emptyset$, $f^{-1}(D) \neq \emptyset$

A is connected!

so we can find a sequence (x_n) in $f^{-1}(C)$ with limit in $f^{-1}(D)$

(x_n) in $f^{-1}(C)$ and limit $\underbrace{\quad}_{y_0}$ in $f^{-1}(D)$
or vice versa

$$\begin{array}{ccc} (x_n) & \longrightarrow & y_0 \\ \cap & & \in f^{-1}(D) \\ f^{-1}(C) & & \end{array}$$

Since f
is continuous

$$\begin{array}{ccc} f(x_n) & \longrightarrow & f(y_0) \\ \cap & & \cap \\ C & & D \end{array}$$

or vice versa