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Tuesday, November 18, 2025 15:15

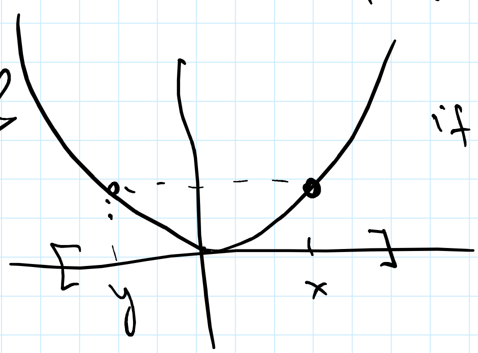
HW6/P3)

$f : [a,b] \to \mathbb{R}$ cont.

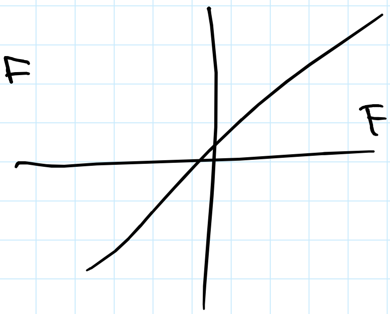
$F = \{ x \in [a,b] \mid \exists\, y \in [a,b],\ y \neq x,\ f(x) = f(y) \}$

Show: $F \neq \emptyset$ or uncountable.

Ex

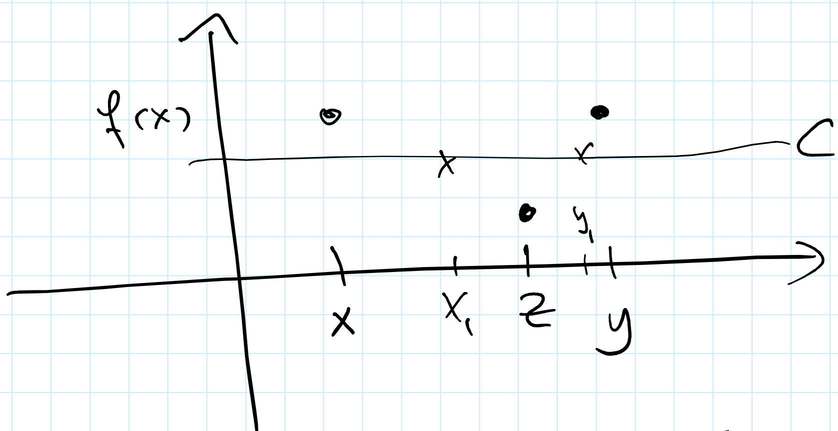


if $x \neq 0 \Rightarrow x \in F$



$F = \emptyset$

suppose $F \neq \emptyset$, say $f(x) = f(y)$ f.s. $y \neq x$



Case 1 $f(x)$ constant on $[x,y]$ ✓

$F \supseteq [x,y] \Rightarrow F$ is unctble.

Case 2 f is not cst. on $[x,y]$

say $\exists\, z \in (a,b) : f(z) \neq f(x)$

Now use IVThm (wlog $x < y$)

given $C \in (f(x), f(z))$ $\quad \exists\, x_1 \in (x,z)$ $\quad f(x_1) = C$

$\exists\, y_1 \in (z,y)$

$\exists\ y_1 \in (z, y)$

$f(y_1) = c$

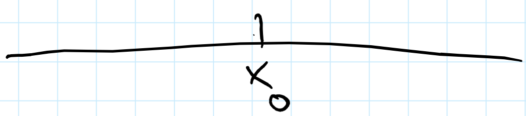
I can do for an uncble # of c's,

gives me an uncble # of pairs $x_1, y_1 \in F$

so F is uncountable.

Ex. 4.35

If x_0 is an isolated point of $A \subset \mathbb{R}$ then any $f : A \to \mathbb{R}$ is continuous at x_0



there is a $\delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \cap A = \{x_0\}$

given $\varepsilon > 0$, let $\delta > 0$ be as above

whenever $|x - x_0| < \delta$ for some $x \in A$

then $x = x_0$ and then

$f(x_0) = f(x)$

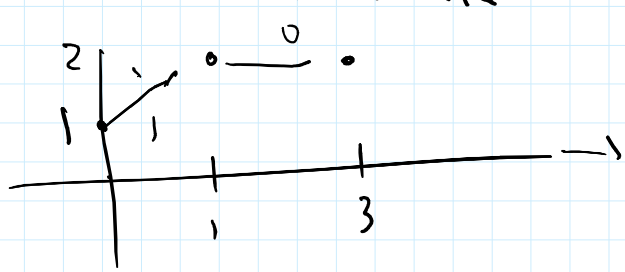
$|f(x_0) - f(x)| = 0 < \varepsilon$

Ex. 5.3.3

$h : [0,3] \to \mathbb{R}$ dfb

$h(0) = 1$, $h(1) = 2$, $h(3) = 2$

Argue that $h'(x) = \frac{1}{4}$ at some pt. in the domain



MVThm find a line-segment

MVThm find a line-segment b/w two points on the graph of the function s. that its slope is $1/4$

Can't do that:
have to show that derivatives have the IVP.

X (Darboux's theorem)

Ex. 5.2.2 ? f, g are not dfb at 0, but $f \cdot g$ is dfb at 0

$f(x) = g(x) = |x|$ not dfb at 0

$f(x) \cdot g(x) = x^2$ dfb at 0

$|x|^2 = x^2$ for $x \in \mathbb{R}$

Ex 2.6.4

(a_n) Cauchy sequence

$(c_n) = (-1)^n a_n$

? (c_n) a Cauchy sequence?

NO $a_n = 1 \ \forall n \in \mathbb{N}$ Cauchy sequ.

$c_n = (-1)^n$ not Cauchy sequence.

Is $f(x) = x \sin \frac{1}{x}$ uniform cont. on $(0,1)$?

$$g(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$g(x)$ is continuous on $[0,1]$

$\Rightarrow g(x)$ is uniform cont. on $[0,1]$

$\Rightarrow g(x)$ is uniformly cont. on $[0,1]$

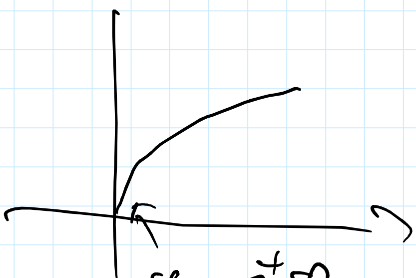
$\Rightarrow g(x)$ is uniformly cont. on $(0,1)$

Since $g = f$ on $(0,1)$

f unif'ly cont. on $(0,1)$

Aside $\quad f:[0,1] \to \mathbb{R}$

$f(x) = \sqrt{x}$



f cont on $[0,1]$, therefore unif'ly cont on $[0,1]$

True or false $\longrightarrow$ $g: \mathbb{R} \to \mathbb{R}$ cont. on $\mathbb{R}$

$g(x) \geq 0 \ \forall x < 1$

then $g(1) \geq 0$

take a sequence $(x_n) \in \mathbb{R}$ $\quad x_n \leq 1 \ \forall n$

and $x_n \to 1$

Since g is cont. at 1,

we get $\underbrace{g(x_n)}_{\geq 0} \longrightarrow g(1)$

$\Rightarrow \ \geq 0$ ✓

What if instead

$g(x) > 0 \quad \forall x < 1$

$\overset{?}{\Rightarrow} g(1) > 0$

Example $\quad g(x) = 1 - x$

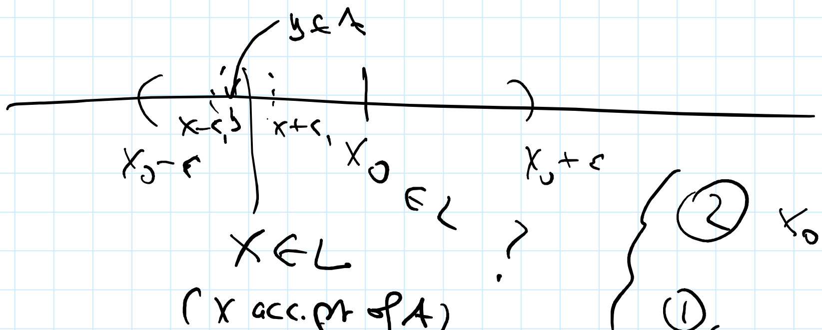
Ex 8.27 Given $A \subseteq \mathbb{R}$, L be the set of acc. pts of A. Show that L is closed.

pf. Let x_0 be an acc. pt of L.

We have to show: $x_0 \in L$

i.e. x_0 is an acc. pt of A.

Let $\varepsilon > 0$. We know $\exists\, x \in L$, $x_0 \neq x$ s.that $|x - x_0| < \varepsilon$



② $x_0 \notin (x - \varepsilon_1, x + \varepsilon_1)$

There is an $\varepsilon_1 > 0$ s.that ① $(x - \varepsilon_1, x + \varepsilon_1) \subseteq (x_0 - \varepsilon, x_0 + \varepsilon)$

$(x - \varepsilon_1, x + \varepsilon_1) \cap A \neq \emptyset$,

say $y \in (x - \varepsilon_1, x + \varepsilon_1) \cap A$

$y \neq x_0$, $y \in (x_0 - \varepsilon, x_0 + \varepsilon)$

so x_0 is an acc. pt of A

and thus $x_0 \in L$

If f is diff at x_0, f is also cont. at x_0

pf

$$\Delta f(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f(x) - f(x_0) = \Delta f(x) \cdot (x - x_0)$$

then $\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \Delta f(x) \cdot \lim_{x \to x_0} (x - x_0)$

$$\lim_{x \to x_0} (f(x) - f(x_0)) = f'(x_0) \cdot 0$$

i.e. $\lim_{x \to x_0} f(x) = f(x_0)$