Limes Inferior and Limes Superior

Let (a_n) be a bounded sequence of real numbers. We define the LIMES INFERIOR[1] and LIMES SUPERIOR of the sequence as

$$\liminf_{n\to\infty} a_n := \lim_{k\to\infty} \left(\inf\{a_n \mid n \geq k\}\right),$$

and

$$\limsup_{n\to\infty} a_n := \lim_{k\to\infty} \left(\sup\{a_n \mid n \geq k\}\right).$$

Problem 1

Explain why the numbers $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ are well-defined[2] for every bounded sequence (a_n).

One can define the notions of lim sup and lim inf without knowing what a limit is:

Problem 2

Show that the limes inferior and the limes superior can also be defined as follows:

$$\liminf_{n\to\infty} a_n := \sup\left\{\inf\{a_n \mid n \geq k\} \mid k \in \mathbb{N}\right\},$$

and

$$\limsup_{n\to\infty} a_n := \inf\left\{\sup\{a_n \mid n \geq k\} \mid k \in \mathbb{N}\right\}.$$

Problem 3

Show that a bounded sequence (a_n) converges if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

[1] "limes" means limit in Latin.

[2] An object is well-defined if it exists and is uniquely determined.

Problem 4

Let (a_n) be a bounded sequence of real numbers. Show that (a_n) has a subsequence that converges to $\limsup_{n\to\infty} a_n$.

Problem 5

Let (a_n) be a bounded sequence of real numbers, and let (a_{n_k}) be one of its converging subsequences. Show that

$$\liminf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$