Monotone Functions

Let $a < b$ be real numbers. A function $f : [a, b] \to \mathbb{R}$ is called INCREASING on $[a, b]$, if $x < y$ implies $f(x) \leq f(y)$ for all $x, y \in [a, b]$. It is called STRICTLY INCREASING on $[a, b]$, if $x < y$ implies $f(x) < f(y)$ for all $x, y \in [a, b]$.

Similarly, a function $f : [a, b] \to \mathbb{R}$ is called DECREASING on $[a, b]$, if $x < y$ implies $f(x) \geq f(y)$ for all $x, y \in [a, b]$. It is called STRICTLY DECREASING on $[a, b]$, if $x < y$ implies $f(x) > f(y)$ for all $x, y \in [a, b]$.

A function $f : [a, b] \to \mathbb{R}$ is called MONOTONE on $[a, b]$ if it is increasing on $[a, b]$ or it is decreasing on $[a, b]$.

As we have seen in the last section, a function can fail to have limits for various reasons. Monotone functions, on the other hand, are easier to understand: a monotone function fails to have a limit at a point if and only if it "jumps" at that point. The next task makes this precise.

Problem 1

Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be an **increasing** function. Let $x_0 \in (a, b)$. We define

$$L(x_0) = \sup\{f(y) \mid y \in [a, x_0)\}$$

and

$$U(x_0) = \inf\{f(y) \mid y \in (x_0, b]\}$$

Show: $f(x)$ has a limit at x_0 if and only if $U(x_0) = L(x_0)$. In this case

$$U(x_0) = L(x_0) = f(x_0) = \lim_{x \to x_0} f(x).$$

Problem 2

Under the assumptions of the previous task, state and prove a result discussing the existence of a limit at the endpoints a and b.

Problem 3
Let $a < b$ be real numbers and let $f : [a, b]$ be an increasing function. Show that the set

$$\{y \in [a, b] \mid f(x) \text{ does not have a limit at } y\}$$

is finite or countable[1].

You may want to show first that the set

$$D_n := \{y \in (a, b) \mid (U(y) - L(y)) > 1/n\}$$

is finite for all $n \in \mathbb{N}$.

Let us look at an example of an increasing function with countably many "jumps": Let $g : [0, 1] \to [0, 1]$ be defined as follows:

$$g(x) = \begin{cases} 0 & \text{, if } x = 0 \\ \dfrac{1}{n} & \text{, if } x \in \left(\dfrac{1}{n+1}, \dfrac{1}{n}\right] \text{ for some } n \in \mathbb{N} \end{cases}$$

Figure 1 on the following page shows the graph of $g(x)$. Note that the function is well defined, since

$$\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right] = (0, 1],$$

and

$$\left(\frac{1}{m+1}, \frac{1}{m}\right] \cap \left(\frac{1}{n+1}, \frac{1}{n}\right] = \emptyset$$

for all $m, n \in \mathbb{N}$ with $m \neq n$.

Problem 4
Show the following:

1. The function $g(x)$ defined above fails to have a limit at all points in the set $D := \left\{\dfrac{1}{n} \mid n \in \mathbb{N}\right\}$.

2. The function $g(x)$ has a limit at all points in the complement $[0, 1] \setminus D$.

[1] A set is called COUNTABLE, if all of its elements can be arranged as a sequence $y_1, y_2, y_3, \ldots$ with $y_i \neq y_j$ for all $i \neq j$.

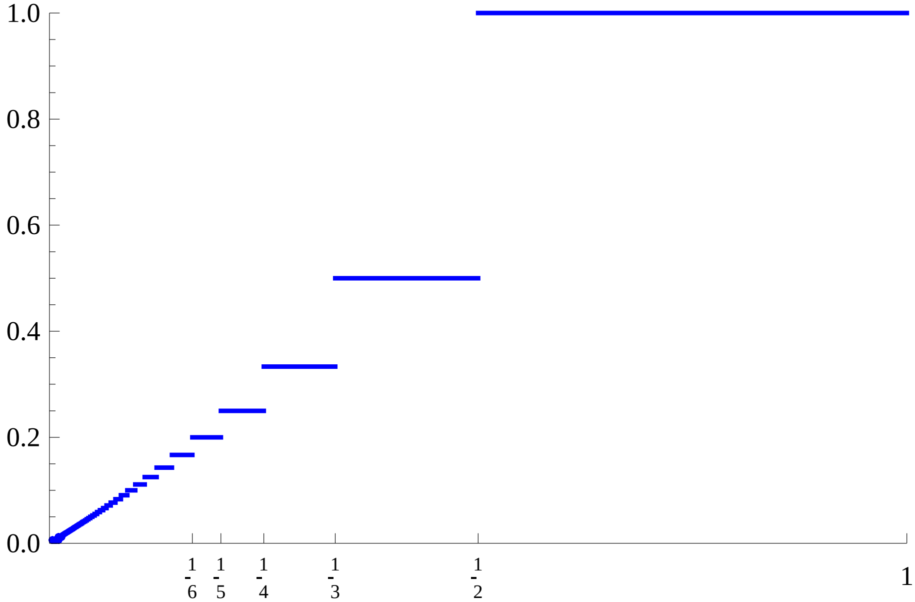

Figure 1: The graph of a function with countable many “jumps”