A differential equation is an equation involving derivatives

$$y'(x) = x$$

x independent variable
y dependent variable

Solving a DEQ means to find functions $y(x)$ that solve the equation.

✓ $y(x) = \frac{x^2}{2}$ is a sol. of $y'(x) = x$

✗ ? $y(x) = \cos x$
check: $y'(x) = -\sin x \neq x = x$

? $y(x) = \frac{x^2}{2} + C$
check: $y'(x) = x$ ✓

→ we can check solutions
→ DEQ have many solutions
→ Finding solutions may involve integration.
→ Solving DEQs is a generalization of "techniques of integration"

of "techniques of integration"

Example 2 $\quad y'(x) = y(x)$

✓? $y(x) = e^x \qquad y'(x) = e^x = y(x)$ ✓

✗? $y(x) = e^x + C \qquad y'(x) = e^x \neq y(x) = e^x + C$

✓ $y(x) = 0$

✗? $y(x) = \ln x \qquad y'(x) = \frac{1}{x} \neq \ln x$

✗? $y(x) = e^{-x} \qquad y'(x) = -e^{-x} \neq e^{-x} = y(x)$

✓? $y(x) = e^{x+C} \qquad y'(x) = e^{x+C} = y(x)$

✓ $y(x) = Ke^x \qquad e^{x+C} = e^x e^C = e^x K$

Example $\quad y'(x) = 2 \cdot y(x)$

✓? $y(x) = e^{2x} \qquad y'(x) = 2e^{2x} = 2 \cdot y(x)$ ✓

✓ $y(x) = Ce^{2x} \qquad y'(x) = C\,2e^{2x} = 2Ce^{2x} = 2y(x)$

is called the general solution to the DEQ

Example $\quad y'(x) = -y(x)$

✓ $y(x) = Ce^{-x} \qquad y'(x) = C(-e^{-x})$

$\checkmark\ y(x) = Ce^{-x}$ $\quad \checkmark\ y'(x) = C(-e^{-x}) = -y(x)$

Initial Value Problem (IVP)

$$\begin{cases} y'(x) = -y(x) \\ y(0) = -3 \end{cases}$$

- IVPs usually have unique solutions!

Procedure: Find the general solution; then use the 2nd equation to solve for C

$$y(x) = Ce^{-x}$$

$$-3 = y(0) = C \cdot e^0 = C$$

$$\Rightarrow C = -3$$

Sol. to IVP: $y(x) = -3e^{-x}$

1st order DEQ's contain only the first derivative

2nd order DEQ contains also the second derivative

all our differential equations will be ordinary DEQ

will be ordinary DEQ
= only one independent variable
"ODE"
if we have several independent variables, the DEQ is called a partial DEQ
"PDE"

Example $y''(x) = k$ (k cst)

✓ ? $y(x) = \frac{k}{2}x^2$ $\quad y'(x) = \frac{k}{2} 2x = kx$

$y''(x) = k$

$y(x) = k\frac{x^2}{2}$

? $y(x) = k\frac{x^2}{2} + C_1 x + C_2$

$y'(x) = kx + C_1 + 0$

$y''(x) = k + 0 + 0 = k$ ✓

(N.B if $k = -g$ we're talking about free fall)

IVP $\left.\begin{array}{l} y''(x) = k \\ y(0) = D_1 \\ y'(0) = D_2 \end{array}\right]$

Example $\quad y''(x) = y(x)$

$\checkmark$? $y(x) = e^x \qquad y'(x) = e^x \quad \checkmark$

$y''(x) = e^x = y(x)$

$\checkmark$? $y(x) = Ce^x \qquad y''(x) = Ce^x = y(x)$

$\checkmark$? $y(x) = De^{-x} \qquad y''(x) = D(-(-e^{-x}))$

$= De^{-x} = y(x)$

general solution:

$$y(x) = Ce^x + De^{-x}$$

? $y(x) = \cosh x \qquad y'(x) = \sinh x$

$y''(x) = \cosh x = y(x)$

general solution:

$$y(x) = A\cosh x + B\sinh x$$

(Recall $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$)

Example $\quad y''(x) = -y(x)$

$\checkmark$? $y(x) = \sin x \qquad y'(x) = \cos x$

$y''(x) = -\sin x$

$= -y(x)$

$\checkmark$ $y(x) = C\sin x$

$\checkmark$? $y(x) = D\cos x \qquad y'(x) = -D\sin x$

$y''(x) = -D\cos x$

$$y''(x) = -D\cos x$$
$$= -y(x)$$

general solution:

$$y(x) = C\sin x + D\cos x$$
$$[\, y(x) = Ae^{ix} + Be^{-ix} \,]$$

Example $y''(x) = -2\,y(x)$

$\times$ $y(x) = e^{-2x}$ $y'(x) = -2e^{-2x}$
$$y''(x) = +4e^{-2x}$$
$$\neq -2y(x)$$

$\times$ $y(x) = 2\sin x$ $y'(x) = 2\cos x$
$$y''(x) = -2\sin x$$
$$= -y(x)$$

$$y(x) = \sin(\sqrt{2}x)$$
$$y'(x) = \cos(\sqrt{2}x)\,\sqrt{2}$$
$$= \sqrt{2}\,\cos(\sqrt{2}x)$$
✓
$$y''(x) = \sqrt{2}\,(-\sin(\sqrt{2}x))\cdot\sqrt{2}$$
$$= -2\sin(\sqrt{2}x)$$
$$= -2\,y(x)$$

general solution

$$y''(x) = A\cdot\sin(\sqrt{2}x) + B\cdot\cos(\sqrt{2}x)$$