Recall:

$$y'(t) + 2y(t) = 0$$

$$\Leftrightarrow \quad y'(t) = -2y(t)$$

Solutions: $y(t) = Ce^{-2t}$

check $y'(t) = -2(Ce^{-2t}) = -2y(t)$

Example $y'(t) + 2y(t) = 3 \quad | \cdot e^{2t}$

$$y'(t)e^{2t} + 2e^{2t}y(t) = 3e^{2t}$$

product rule backwards $\rightarrow$ $[y(t) \cdot e^{2t}]' = 3e^{2t}$ (!)

integrate on both sides: $y(t)e^{2t} = \int 3e^{2t} = \frac{3}{2}e^{2t} + C$

$$\boxed{y(t) = \frac{3}{2} + Ce^{-2t}}$$

Why e^{2t}?

$$y'(t) + 2y(t) = 3 \quad | \cdot I(t)$$

$$\underbrace{y'(t)\,I(t) + 2I(t)}\,y(t) = 3\,I(t)$$

Want $(y(t) \cdot I(t))' = 3I(t)$

$$\overset{\shortparallel}{y'(t)I(t) + I'(t)y(t)}$$

need: $I'(t) = 2I(t)$

solutions: $I(t) = Ce^{+2t}$

solution: $I(t) = Ce^{+2t}$

(we only need one solution!)

"integrating factor" $\longrightarrow$ $I(t) = e^{2t}$

$\longrightarrow$ How to find integrating factor in general?

$$y'(t) + p(t)\cdot y(t) = q(t)$$

in the previous case: $p(t) = 2$

$q(t) = 3$

$$y'(t) + p(t)\, y(t) = q(t) \quad \Big| \cdot I(t)$$

$$\underbrace{y'(t)\, I(t) + p(t)\, I(t)\, y(t)}_{} = q(t) \cdot I(t)$$

want $\quad [\, y(t) \cdot I(t) \,]' = \ldots$

$$= y'(t) \cdot I(t) + y(t)\, I'(t)$$

$$I'(t) = p(t) \cdot I(t)$$

$$\frac{I'(t)}{I(t)} = p(t)$$

$$\int \frac{I'(t)}{I(t)}\, dt = \int p(t)\, dt$$

$$\ln(I(t)) = \int p(t)\, dt$$

$$\boxed{(\ln(I))' = \frac{I'}{I}}$$

$$\ln(I(t)) = \int p(t)\,dt$$

I

$$\Rightarrow \boxed{I(t) = e^{\int p(t)\,dt}}$$

in our case $p = 2$ $\quad \int 2\,dt = 2t$

$$I(t) = e^{2t}$$

Exa $\quad y' - 3y = e^{2t}$

$p(t) = -3 \qquad \int p(t)\,dt = \int -3\,dt = -3t$

$$I(t) = e^{-3t}$$

$$y' - 3y = e^{2t} \,\Big|\, \cdot e^{-3t}$$

$$y'e^{-3t} - 3e^{-3t}y = e^{-t}$$

$$\left[y \cdot e^{-3t}\right]' = e^{-t}$$

Check!

$$y'e^{-3t} + y(-3e^{-3t})$$

$$y \cdot e^{-3t} = \int e^{-t}\,dt = -e^{-t} + C$$

$$\Rightarrow \boxed{y = -e^{2t} + Ce^{3t}}$$

Initial Value Problem:

$$y(0) = -5$$

$$-5 = y(0) = -e^{2\cdot 0} + Ce^{3\cdot 0}$$

$$-5 = -1 + C$$

$$-5 = -1 + c$$
$$-4 = c$$

unique solution satisfying the IVP:

$$y(t) = -e^{2t} - 4e^{3t}$$

Exa

$$y'(t) + \frac{y(t)}{t} = 1 \quad \Big| \cdot t$$

$$p(t) = \frac{1}{t} \qquad \int p(t)\,dt = \int \frac{1}{t}\,dt = \ln t$$

$$I(t) = e^{\ln t} = t$$

$$y' \cdot t + y = t$$

Check

$$(y \cdot t)' = t$$

$$y' t + 1 \cdot y = t$$

$$y \cdot t = \int t\,dt = \frac{t^2}{2} + c$$

$$\boxed{y = \frac{t}{2} + \frac{c}{t}}$$

Exa

$$y'(t) - \frac{y(t)}{t} = \cos t$$

$$p(t) = -\frac{1}{t} \qquad \int -\frac{1}{t}\,dt = -\ln t$$

$$I(t) = e^{-\ln t}$$

$$I(t) = e^{-\ln t} = e^{\ln \frac{1}{t}} = \frac{1}{t}$$

$$I(t) = \frac{1}{t}$$

$$\frac{y'(t)}{t} - \frac{y(t)}{t^2} = \frac{\cos t}{t}$$

$$\left(\frac{y(t)}{t}\right)' = \frac{\cos t}{t}$$

$$y'(t)\frac{1}{t} + y(t)\left(-\frac{1}{t^2}\right) = \frac{\cos t}{t}$$

$$\frac{y(t)}{t} = \int \frac{\cos t}{t}\,dt$$

no way to "integrate" $\frac{\cos t}{t}$

$$y(t) = t\int \frac{\cos t}{t}\,dt$$