Separable DEQs

Last time:

Linear DEQ:

$$y' + p(t)\cdot y = q(t)$$

complete success:

can write solutions as $y(t) = \ldots$

solutions in explicit form

Exa 1

$$\frac{dy}{dt} = -\frac{t}{y}$$

separate t's and y's

$$y\,dy = -t\,dt$$

integrate both sides

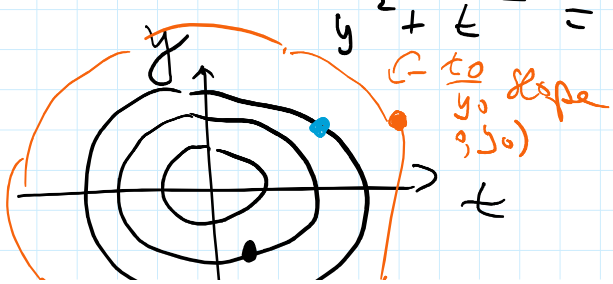
$$\int y\,dy = \int -t\,dt$$

$$\frac{y^2}{2} = -\frac{t^2}{2} + C \qquad C \text{ cst.}$$

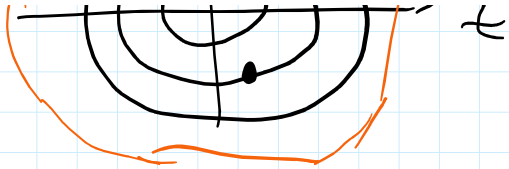
at this point we have solved the DEQ in implicit form

$$\frac{y^2}{2} + \frac{t^2}{2} = C$$

$$y^2 + t^2 = D \qquad (D \text{ cst } D = 2C)$$



$y(t_0) = y_0$



Exa 2

$$\frac{dy}{dt} = \frac{y}{t}$$ this is also a linear DEQ!

$$\frac{dy}{y} = \frac{dt}{t}$$ separated the DEQ ✓

$$\int \frac{dy}{y} = \int \frac{dt}{t}$$

$$\ln|y| = \ln|t| + C$$

solutions in implicit form

$$|y| = e^{\ln|t| + C}$$ C arb.

$$|y| = K e^{\ln|t|}$$ $K = e^C$ $(K > 0)$

$$|y| = K|t|$$

$$y = \pm K|t|$$

$$y = L \cdot t$$ lines thru origin

$$\frac{dy}{dt} = \frac{y}{t}$$

$$\Leftrightarrow \quad y' - \frac{1}{t} \cdot y = 0$$

Solve the lin. DEQ

$$\hookrightarrow y = C \cdot t$$

Exa 3

$$\frac{dy}{dt} = 2y + 1$$

$$\frac{dy}{2y+1} = dt$$ Sep. DEQ

$$\frac{dy}{2y+1} = dt \quad \text{sp. DEQ}$$

$$\int \frac{dy}{2y+1} = \int dt$$

$$\frac{1}{2}\ln|2y+1| = t + C$$

$$\ln|2y+1| = 2t + D \qquad (D = 2C)$$

$$|2y+1| = e^{2t+D} = e^{2t}e^{D}$$

$$|2y+1| = Ke^{2t} \qquad (K>0) \qquad K = e^{D}$$

$$2y+1 = \pm Ke^{2t}$$

$$2y+1 = Le^{2t} \qquad L \text{ arb.}$$

$$2y = Le^{2t} - 1$$

$$y = \frac{Le^{2t}-1}{2}$$

Let's check the solution

$$\frac{dy}{dt} = \frac{2Le^{2t}}{2} = Le^{2t}$$

$$2y+1 = 2\left(\frac{Le^{2t}-1}{2}\right)+1 = Le^{2t} \quad \checkmark$$

Ex 4

$$\frac{dy}{dt} = e^{-y}$$

$$e^{y}\,dy = dt \quad \text{sp. DEQ} \checkmark$$

$$\int e^{y}\,dy = \int dt$$

$$e^{y} = t + C$$

$$\ln(e^{y}) = \ln(t+C)$$

$$\ln(e^y) = \ln(t+C)$$

$$y = \ln(t+C) \leftarrow$$ solutions in explicit form

IVP $y(0) = 4$

$$4 = y(0) = \ln(0+C) = \ln C$$

$$\Rightarrow C = e^4$$

unique solution: $y(t) = \ln(t+e^4)$

$$-1 = y(0) = \ln(0+C)$$

$$\Rightarrow \ln C = -1 \Rightarrow C = e^{-1}$$

unique sol.: $y(t) = \ln(t+e^{-1})$

For which t will this solution be valid?

$\ln(t+e^{-1})$ exist

$\Leftrightarrow t+e^{-1} > 0$

$\Leftrightarrow t > -e^{-1}$

solution "works" only on $(-e^{-1}, \infty)$

Exa 5 $\quad \dfrac{dy}{dt} = y(1-y)$

$$\frac{dy}{y(1-y)} = dt \quad \text{separable ✓}$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

$$1 = A(1-y) + By$$

$$0 \cdot y + 1 = A + (B-A)y$$

$A = 1$

$0 \cdot y + 1 = A \quad + (B-A)y$

$\Rightarrow \quad A = 1$

$B - A = 0 \Rightarrow A = B = 1$

$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$ $\square$

$\frac{dy}{y(1-y)} = dt$

$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int dt$

implicit form

$\ln|y| - \ln|1-y| = t + C$

u-subst
$z = 1-y$
$dz = -1$

$\ln \frac{|y|}{|1-y|} = t + C$

$\left|\frac{y}{1-y}\right| = e^{t+C}$

$\frac{y}{1-y} = Ke^{t}$

$y = Ke^{t}(1-y)$

$y = Ke^{t} - Kye^{t}$

$y + \cancel{y}e^{t} = Ke^{t}$

$y(1+\cancel{e^{t}}) = Ke^{t}$

$y = \frac{Ke^{t}}{1+Ke^{t}}$ ← sol. in explicit form

check! Are these solutions?

check! Are these solutions?

$$\frac{dy}{dt} = \frac{Ke^t}{(1+Ke^t)^2} \qquad y = \frac{(1+Ke^t)-1}{1+Ke^t}$$

$$y = 1 - \frac{1}{1+Ke^t}$$

$$y(1-y) = \left(1 - \frac{1}{1+Ke^t}\right)\left(\frac{1}{1+Ke^t}\right)$$

$$= \frac{1+Ke^t}{(1+Ke^t)^2} - \frac{1}{(1+Ke^t)^2} = \frac{Ke^t}{(1+Ke^t)^2}$$

Lucky Day!

Exa 6

$$\frac{dy}{dt} = \frac{e^t y}{1+y^2}$$

$y=0$ is a solution

$$\left(\frac{1+y^2}{y}\right)dy = e^t dt \qquad \text{Sep. DEQ} ✓$$

$$\int \left(\frac{1}{y}+y\right)dy = \int e^t dt$$

$$\ln|y| + \frac{y^2}{2} = e^t + C$$

sol. in implicit form

$y=0$ not included here

Not possible to solve in explicit form!

IVP $\quad y(0) = -3$

i.e. $t=0$ & $y=-3$

$$\Rightarrow \quad \ln|-3| + \frac{(-3)^2}{2} = e^0 + C$$

$$\text{so } C = \quad C = \ln 3 + \frac{7}{2}$$

unique solution thru $(0,-3)$ in the (t,y) plane place

is $\ln|y| + \frac{y^2}{2} = e^t + \ln 3 + \frac{7}{2}$

Are we missing solutions? "Easy!"

check the ______

May miss constant solutions

$\left(\frac{dy}{dt} = 0 \right)$

Applications

Cooling pizza

oven is 500°. Take pizza out of the oven and it starts cooling

How long do we have to wait until the pizza can be eaten?

Physics Newton's Law of Cooling

the rate of cooling is

proportional to the difference b/w the object's temperature and the ambient temperature

$C(t)$ temp of pizza at time t ($^\circ F$)

t minutes

C^* ambient temperature $= 70^\circ$

$$\frac{d\,C(t)}{dt} \sim C^* - C(t)$$

$$\Leftrightarrow \boxed{\frac{d\,C(t)}{dt} = k\,(C^* - C(t))}$$

$$\Leftrightarrow \boxed{\frac{d\,C(t)}{dt} = k\left(C^{\infty} - C(t)\right)}$$

Newton's Law of Cooling

$$\left[\begin{array}{l} C(0) = 500 \\ \text{need one more piece of information} \\ C(1) = 480 \end{array}\right.$$

edible temperature; $\bar{C} = 110^{\circ}$

$$\left[\begin{array}{l} C' = k\,(70 - C) \\ C(0) = 500 \\ C(1) = 480 \end{array}\right.$$

this is a linear DEQ!

$$c' + kC = 70k \quad \Big| \cdot e^{kt}$$

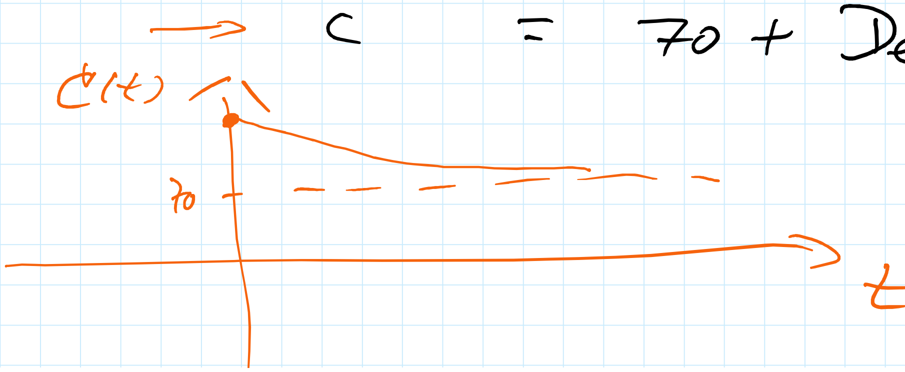
$$c'e^{kt} + ke^{kt}C = 70ke^{kt}$$

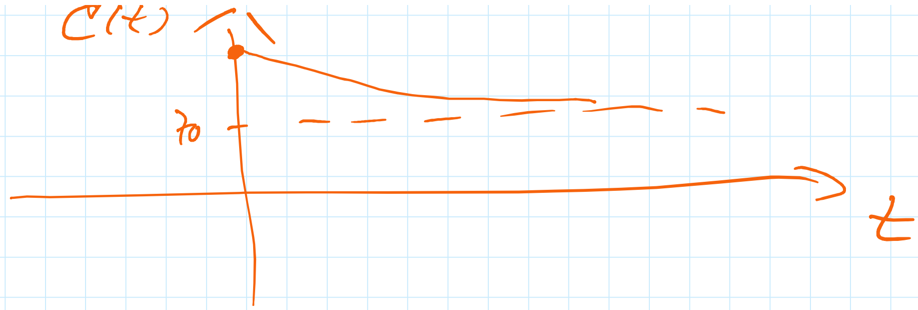
$$(c \cdot e^{kt})' = 70ke^{kt}$$

$$Ce^{kt} = \int 70ke^{kt}\,dt$$

$$Ce^{kt} = 70e^{kt} + D$$

$$\rightarrow \quad C = 70 + De^{-kt}$$




C(t)
70
t