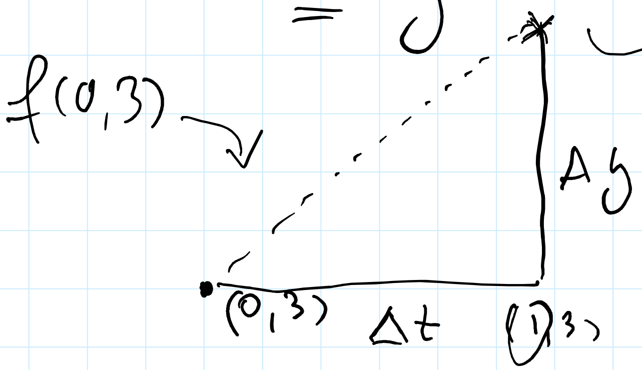
# Euler's Method

→ numerical approximation to an initial value problem

Exa $y' = \underbrace{t - 2y}_{f(t,y)}$ $\quad t_0 = 0$, $y_0 = 3$, $\Delta t = 1$




$$\frac{\Delta y}{\Delta t} \approx f(0,3)$$

$$\Delta y \approx f(0,3) \cdot \Delta t$$

$$= f(0,3) \cdot \Delta t$$

$$t_1 = t_0 + \Delta t = 0 + 1 = 1$$

$$y_1 = y_0 + \Delta y = y_0 + f(t_0, y_0) \cdot \Delta t$$
$$= y_0 + f(0,3) \cdot 1$$
$$= y_0 + (-6) \cdot 1$$
$$= 3 - 6 = -3$$

$\Delta t$ "time-step"

$$t_1 = 1, \quad y_1 = -3$$

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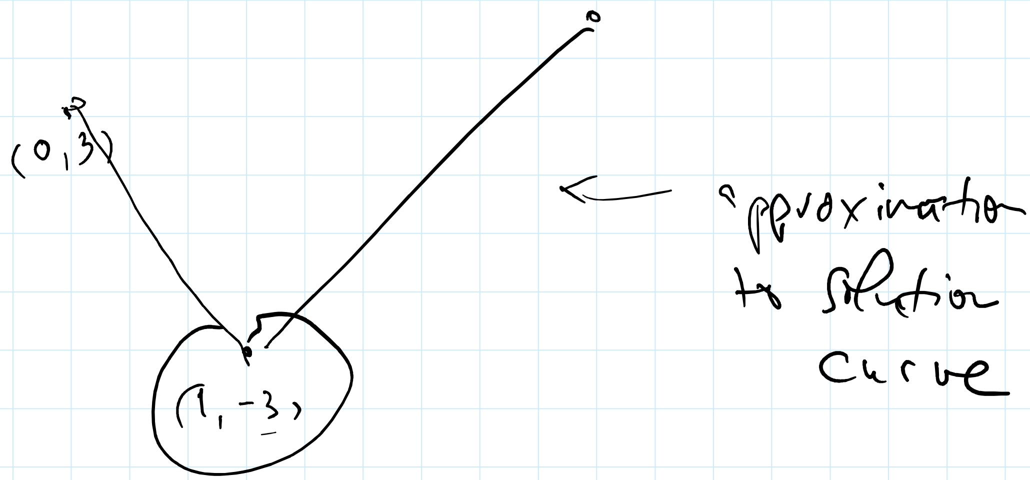
$$t_2 = t_1 + \Delta t = 1 + 1 = 2$$

$$y_2 = y_1 + \Delta y = y_1 + f(t_1, y_1) \cdot \Delta t$$
$$= -3 + 7 \cdot 1 = 4$$

$f(t_1, y_1)$
$= t - 2y$
$= 1 - 2(-3)$
$= 7$

$$t_2 = 2 \quad y_2 = 4$$




Example : $y' = t^2 y + 1$ $\quad t_0 = 0$, $y_0 = 1$

$f(t,y) = t^2 y + 1$ $\quad \Delta t = 1$

$t_0 = 0$
$y_0 = 1$

$t_1 = t_0 + \Delta t = 0 + 1 = 1$

$y_1 = y_0 + \Delta y = y_0 + f(t_0, y_0) \cdot \Delta t$

$= 1 + 1 \cdot 1 = 2$

$t_1 = 1$
$y_1 = 2$

$t_2 = t_1 + \Delta t = 2$

$y_2 = y_1 + \Delta y = y_1 + f(t_1, y_1) \cdot \Delta t$

$2 + 3 \cdot 1 = 5$

$f(t,y)$
$t^2 \cdot y + 1$

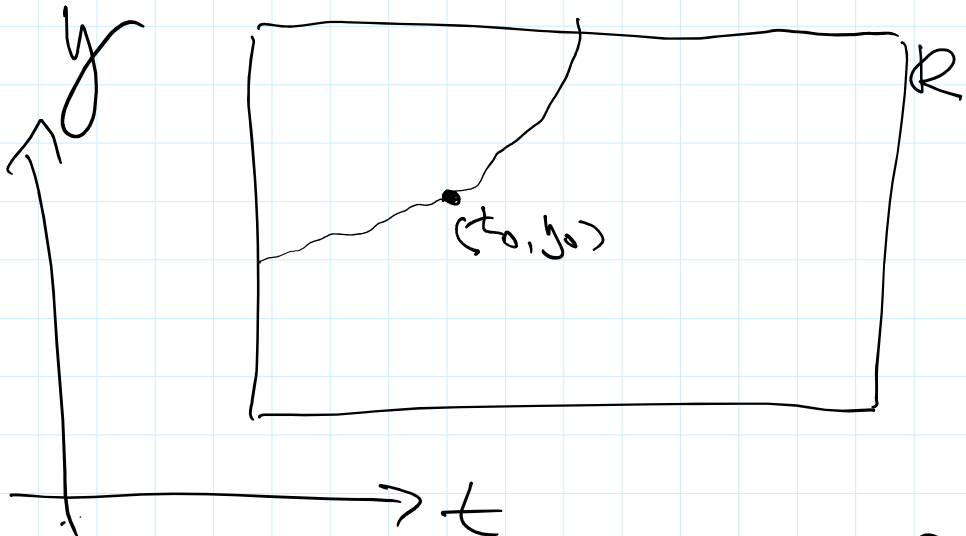
$f = t^2 \cdot y + 1$

$f(1,2) = 3$

$2 + 3 \cdot 1 = 5$

$t_2 = 2$
$y_2 = 5$

Existence and Uniqueness of Solutions




DEQ:

$y' = f(t,y)$

$y(t_0) = y_0$

Theorem: If $f$ and $\frac{\partial f}{\partial y}$ are continuous on $R$, then the IVP will have a unique solution as long as the solutions stays in the rectangle

Example $y' = t^2 y$

$f(t,y) = t^2 y$ continuous on $\mathbb{R} \times \mathbb{R}$

$\frac{\partial f}{\partial y}(t,y) = t^2$ — ,, —

$f(t,y) = \sin(t^2 + y^2)$

Exg $f(t,y) = \sin(t^2+y^2)$

$$\frac{\partial f}{\partial y} = \cos(t^2+y^2)\cdot 2y$$

$$\frac{\partial f}{\partial t} = \cos(t^2+y^2)\cdot 2t$$

$\frac{\partial f}{\partial y}(t,y) = t$ — " —

hold $t$ constant and take derivative w.r.t. $y$

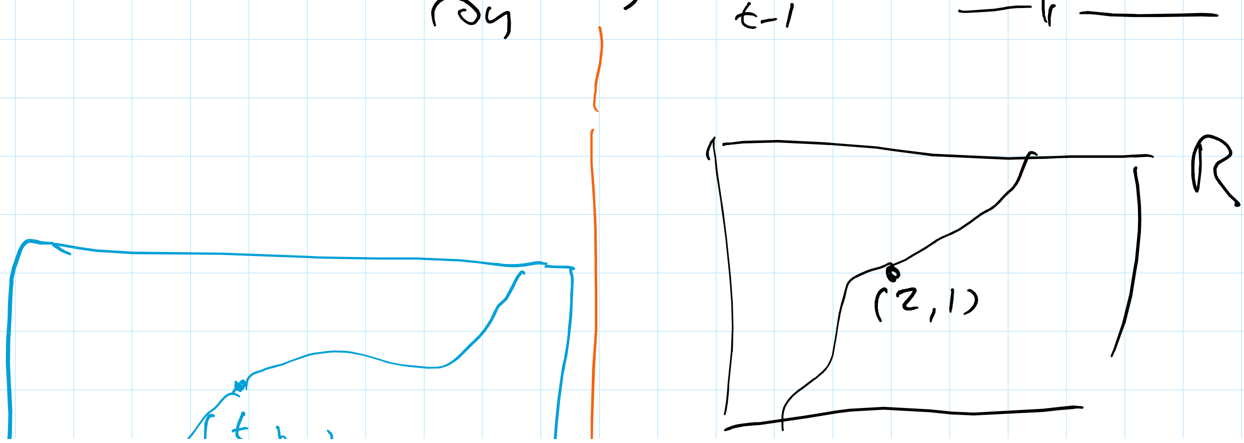

Example

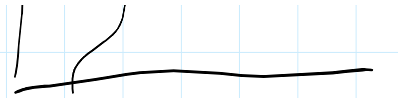
$$y' = \frac{y}{t-1}$$

$$y_0 = 1, \quad t = 2$$

$$f(t,y) = \frac{y}{t-1}$$ cont. except when $t = 1$

$$\frac{\partial f}{\partial y}(t,y) = \frac{1}{t-1}$$ — " —



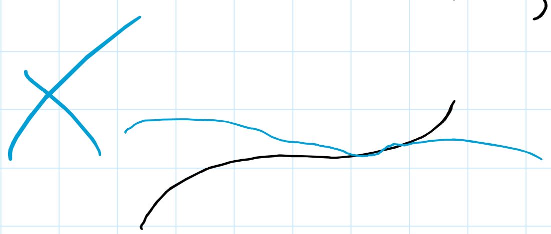



Suppose $f$ and $\frac{\partial f}{\partial t}$ are continuous everywhere

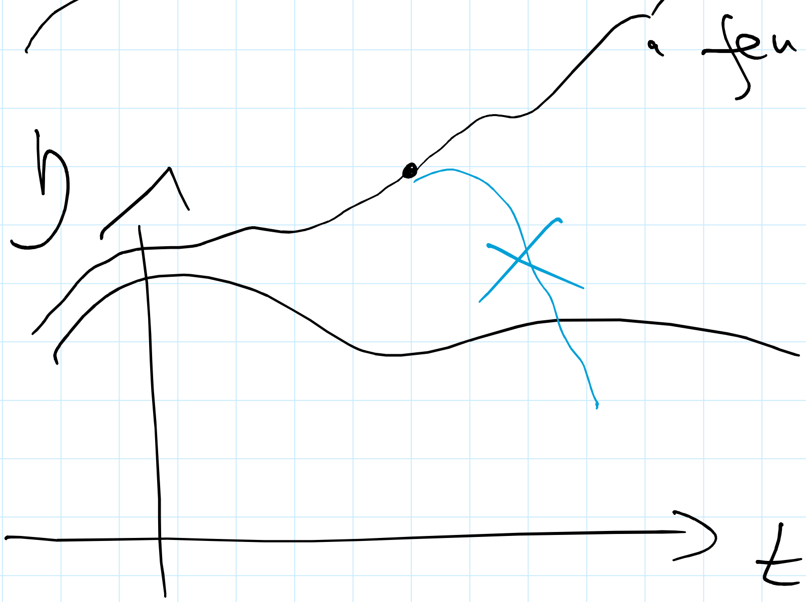
what can we say about solutions to $y' = f(t, y)$ ?

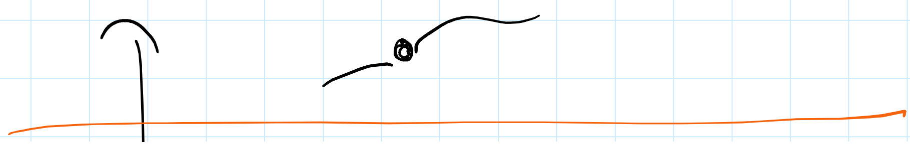
— we can solve all IVPs locally

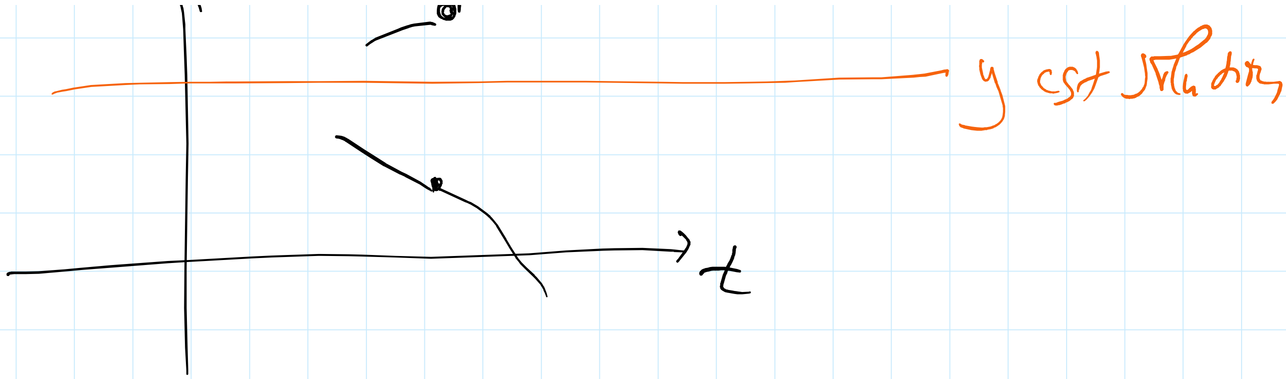
— solutions will never intersect



Every solution builds a fence for all other solutions






## Toricelli's Law


h(t) is the height of water in the cylindrical bucket

how does the height of the water change over time.

$$h'(t) = -2\sqrt{h(t)}$$

lin? ✗ sep? ✓

$$\frac{dh}{dt} = -2\, h^{\frac{1}{2}}(t)$$

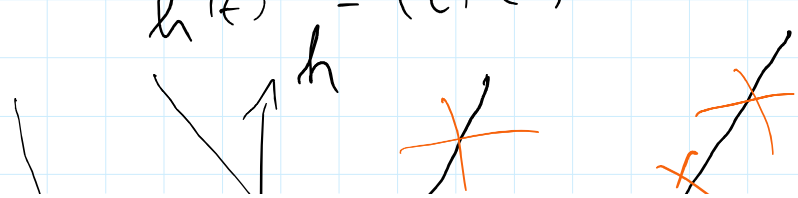
$$-\frac{1}{2} h^{-\frac{1}{2}} \, dh = dt$$

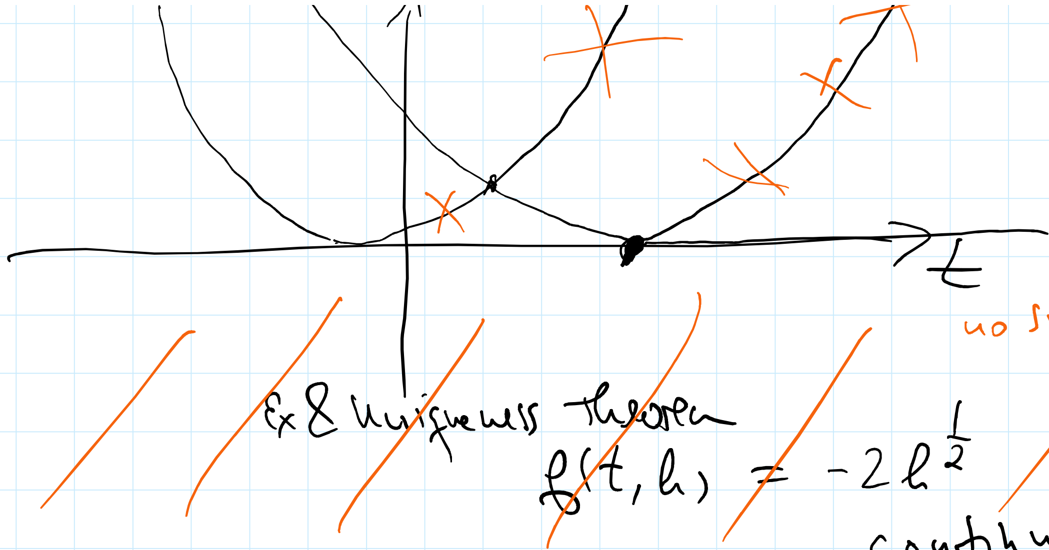
$$-\frac{1}{2}\int h^{-\frac{1}{2}} \, dh = t + C$$

$$-\, h^{\frac{1}{2}} = t + C$$

Solve for h:

$$h(t) = (t+c)^2$$







no solutions here ?

Ex & Uniqueness theorem

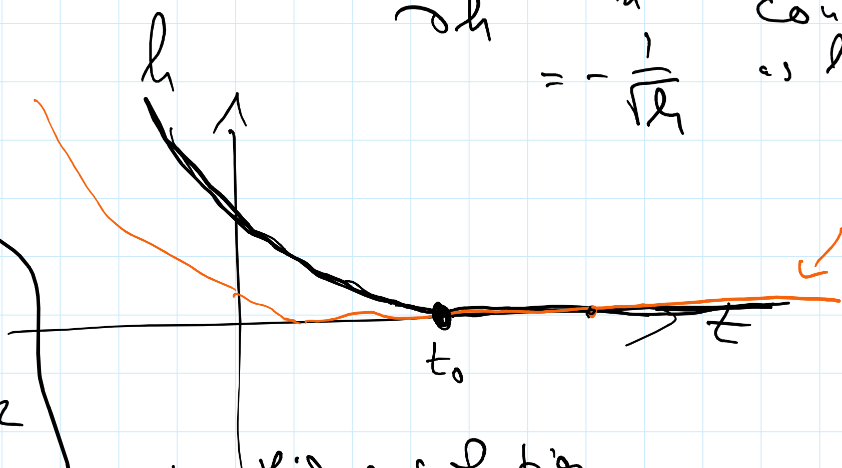
$f(t,h) = -2h^{\frac{1}{2}}$

continuous as long as $h \geq 0$

$\frac{\partial f}{\partial h} = -h^{-\frac{1}{2}}$

$= -\frac{1}{\sqrt{h}}$

continuous as long as $h > 0$





no uniqueness of solutions

$t_0 = \cancel{t_0}$

$y_0 = 0$

$y = (t+c)^2$

$0 = (t_0 + c)^2$

$t_0 + c = 0$

$t_0 = -c$

Is this a solution

$$h(t) = \begin{cases} (t_0 - t)^2, & t \leq t_0 \\ 0, & t > t_0 \end{cases}$$

continuous at $t_0$ ? ✓

does $h'$ exist at $t_0$ ? ✓

$h'(t) = -2(t_0 - t)$ when $t \leq t_0$

$\to 0$ as $t \to t_0$