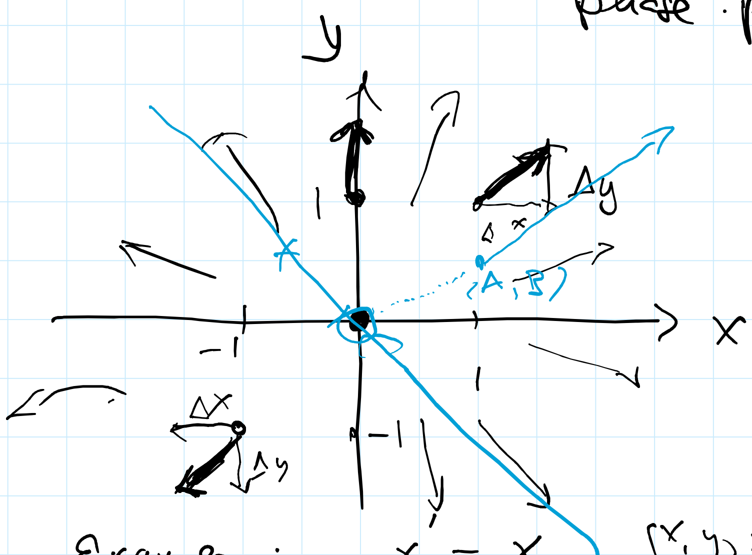
$x'(t) = f(x,y)$ ← autonomous DEQ
$y'(t) = g(x,y)$

phase portraits



phase plane

Example : $x' = x$ $\quad (x,y) = (1,1)$
$y' = y$ $\quad (x,y) = (0,1)$
$(x,y) = (-1,-1)$

vectorfield → lengths varies
directionfield → all vectors have the same length

equil. pts : $x' = 0$, $y' = 0$
in our case $x = 0$, $y = 0$

all solutions move away linearly from the equilibrium point.

$x' = x$ completely decoupled system
$y' = y$

$x' = x$
$y' = y$

completely decoupled system (no interaction b/w the two variables)

$x' = x \Rightarrow \quad x = Ae^t$

$y' = y \Rightarrow \quad y = Be^t$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^t$

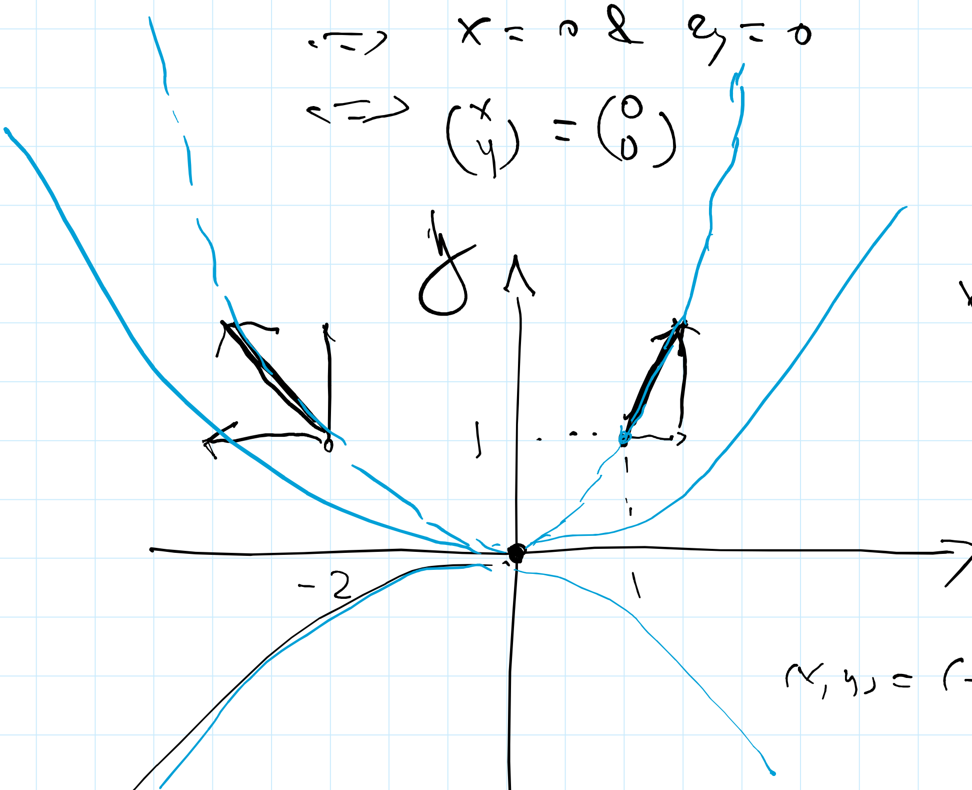
initial conditions
$x(0) = A$
$y(0) = B$

<u>Example</u> $\quad x' = x$
$\quad y' = 2y$

eq pts: $\quad x' = 0 \;\&\; y' = 0$

$\Leftrightarrow \quad x = 0 \;\&\; 2y = 0$

$\Leftrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x' = x$
$y' = 2y$

$(x, y) = (-2, 1)$

$x' = x \quad \Rightarrow \quad x = Ae^t$

$y' = 2y \quad \Rightarrow \quad y = Be^{2t}$

we can write y in terms of x

$x = Ae^t \Longleftrightarrow y = \left(\frac{B}{A^2}\right) \cdot A^2 e^{2t} = \left(\frac{B}{A^2}\right) x^2$

$$x = Ae^{t} \Longleftrightarrow y = \left(\frac{B}{A^2}\right) \cdot A e^{ct} = \left(\frac{B}{A^2}\right) x^2$$

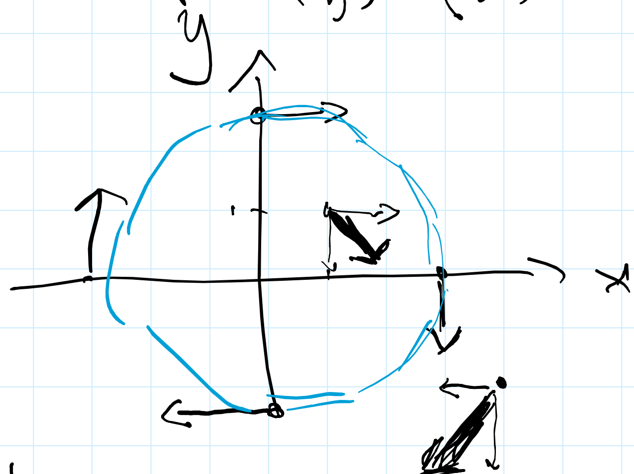
$$y = \left(\frac{B}{A^2}\right) x^2$$

Example

$$x' = y$$
$$y' = -x$$

not decoupled

equil. pt: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x' = y$
$y' = -x$

$(x, y) = (1, 1)$
$(x, y) = (2, 0)$
$(x, y) = 2, -1)$

as it turns out we get circles centered at the equil. pt.

Predator - Prey - Problem

$$r' = 2r\left(1 - \frac{r}{5}\right) - rf$$

$$f' = -f + 2rf$$

equil. pts: $r' = 0$ and $f' = 0$

lazy man's rule

$f' = 0 \Longleftrightarrow \quad -f + 2rf = 0$

$$f' = 0 \iff -f + 2rf = 0$$

$$\iff f(2r-1) = 0$$

Case 1 $f = 0$

Case 2 $r = \frac{1}{2}$

Case 1 $f = 0$

$$r' = 2r\left(1 - \frac{r}{5}\right) = 0$$

Case 1A $r = 0$

Case 1B $r = 5$

equilibrium points:

$(r,f) = (0,0)$

$(r,f) = (5,0)$

Case 2 $r = \frac{1}{2}$

$$0 = r' = 2r\left(1 - \frac{r}{5}\right) - rf$$

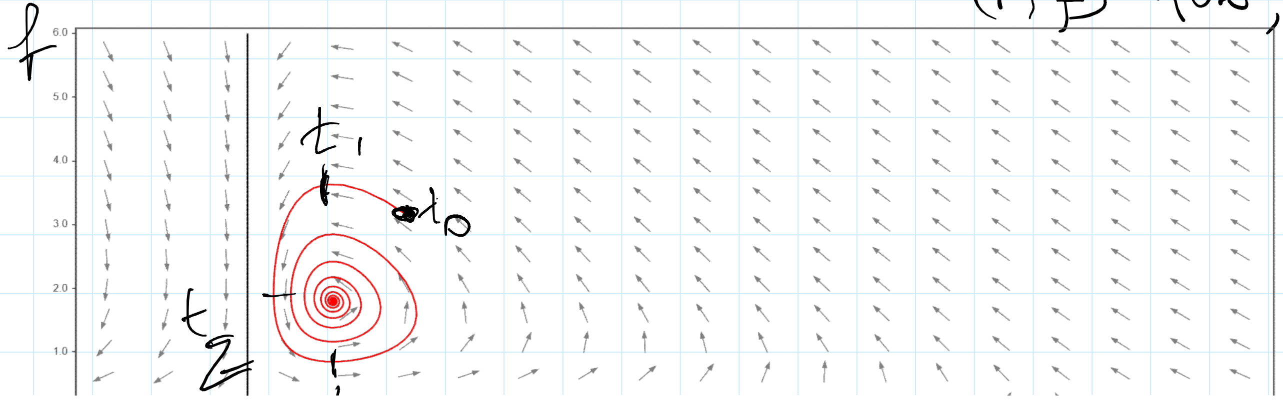
$$0 = \left(1 - \frac{1}{10}\right) - \frac{1}{2}f$$

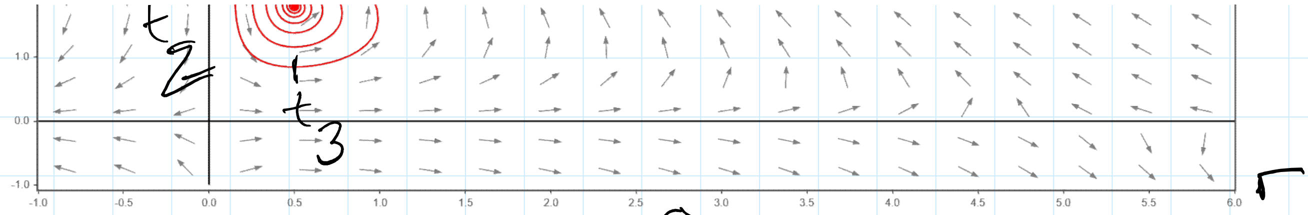
$$0 = \frac{9}{10} - \frac{1}{2}f$$

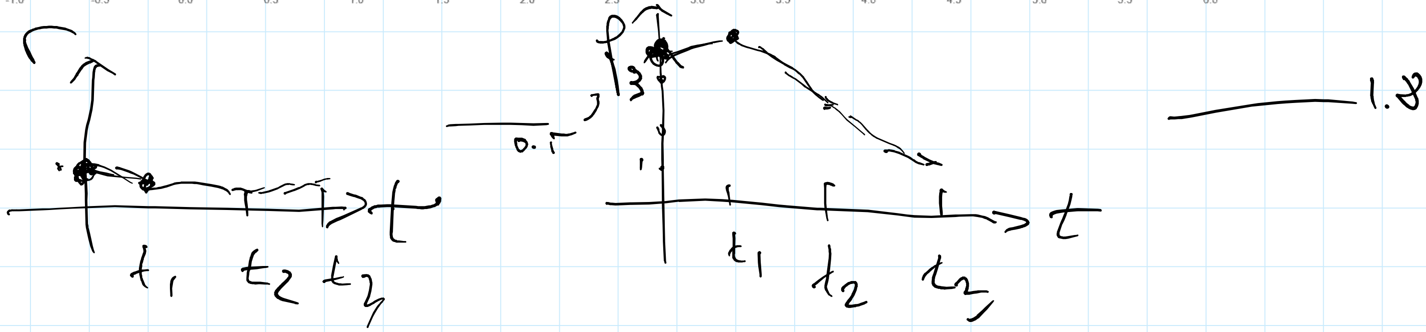
$$\iff \frac{9}{10} = \frac{f}{2} \iff f = \frac{18}{10} = 1.8$$

3rd equil. pt:

$(r,f) = (0.5, 1.8)$







Partially decoupled system

$$x' = 2x - 8y^2$$

$$y' = -3y$$

Solve decoupled DEQ first!

$$y' = -3y \Rightarrow y = Ae^{-3t}$$

$$x' = 2x - 8(Ae^{-3t})^2$$

$$x' = 2x - 8A^2e^{-6t}$$

$$x' - 2x = -8A^2e^{-6t}$$

Integrating factor: e^{-2t}

$$x'e^{-2t} - 2e^{-2t}x = -8A^2e^{-8t}$$

$$(xe^{-2t})' = -8A^2e^{-8t}$$

$$(xe^{-2t})' = -8A^2e^{-8t}$$

$$xe^{-2t} = A^2e^{-8t} + B$$

$$x = A^2e^{-6t} + Be^{2t}$$

$$x(t) = A^2e^{-6t} + Be^{2t}$$
$$y(t) = Ae^{-3t}$$

✓ not A^2!

$$x(t) = 3e^{-6t} + 2e^{2t}$$
$$y(t) = 2e^{-3t}$$
$$A = 2$$

will not be a solution!

example:

$$x' = 2x$$
$$y' = x^3 + y$$

partially decoupled system

equil pts:

$x' = 0 \Rightarrow x = 0$

$\Rightarrow$ $0 = y' = y$

$\Rightarrow y = 0$

$$x' = 2x \Rightarrow x = Ae^{2t}$$

$$y' = (Ae^{2t})^3 + y$$

$$y' - y = A^3e^{6t}$$

Integrating factor: e^{-t}

$$y'e^{-t} - ye^{-t} = A^3e^{5t}$$

$$(ye^{-t})' = A^3e^{5t}$$

$$ye^{-t} = \frac{1}{5}A^3e^{5t} + B$$

$$y = \frac{1}{5}A^3e^{6t} + Be^{t}$$

$$x = Ae^{2t}$$

$$\begin{cases} x = Ae^{2t} \\ y = \frac{1}{5}A^3e^{6t} + Be^{t} \end{cases}$$

$\underline{IVC}$ $\quad x(0) = 1 \qquad 1 = x(0) = Ae^{2\cdot 0} = A$

$\quad y(0) = 2 \qquad \Rightarrow A = 1$

C_{II}

$2 = y(0) = \frac{1}{5} + B$

$\Rightarrow B = \frac{10}{5} - \frac{1}{5} = \frac{9}{5}$

$B = \frac{9}{5}$