2^{nd} order Differential Equations
– this is a special case of systems!

$$x'' = f(x, x', t)$$

system
Introduce a new variable y

$$\left.\begin{aligned} x' &= y \\ y' (= x'') &= f(x, y, t) \end{aligned}\right\}$$ system with variables x and y

Example van der Pol Equation

$$x'' - (1 - x^2)x' + x = 0$$

(autonomous example)

$$x'' = (1 - x^2)x' - x$$

$$\left.\begin{aligned} x' &= y \\ y' &= (1 - x^2)y - x \end{aligned}\right\}$$ system

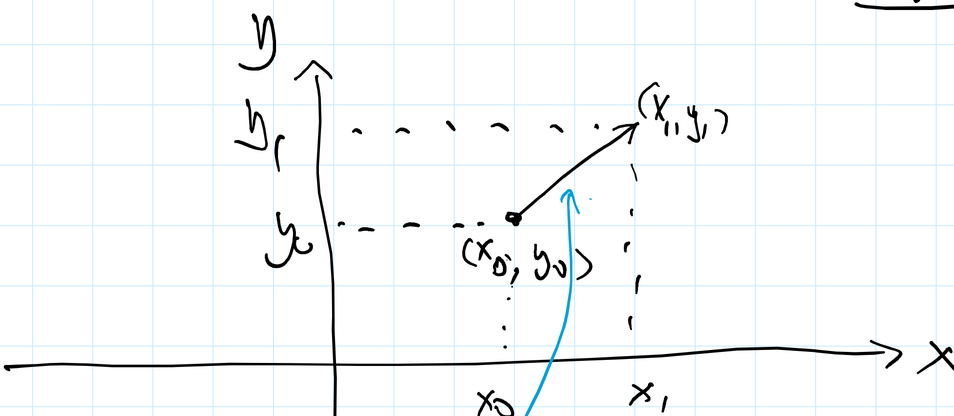
Convert a non-autonomous 1^{st} order DEQ into an autonomous system:

$$x' = f(x, t)$$

$$\left.\begin{aligned} t' &= 1 \\ x' &= f(x, t) \end{aligned}\right\}$$ autonomous system

The van-der-Pol equation exhibits an attractive limit cycle!

Euler Approximations – in the phase plane



direction

$(x'(x_0,y_0),\ y'(x_0,y_0))$

$= (f(x_0,y_0),\ g(x_0,y_0))$

$$x' = f(x,y)$$
$$y' = g(x,y)$$

time increment : Δt

$$\boxed{\begin{aligned} x_1 &= x_0 + f(x_0,y_0)\,\Delta t \\ y_1 &= y_0 + g(x_0,y_0)\cdot \Delta t \end{aligned}}$$

Example

$$x' = 2y \qquad x_0 = 2$$
$$y' = -x \qquad y_0 = 3$$

Initial value problem

$$\Delta t = 1$$

$x_0 = 2$

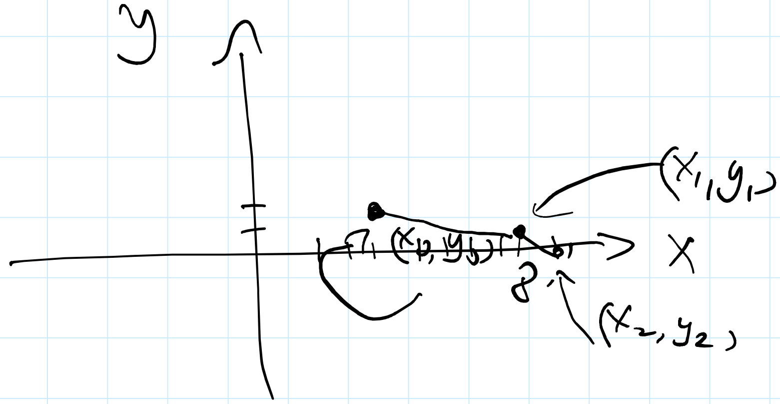
$y_0 = 3$

$x_0 = 2$

$y_0 = 3$

$$x_1 = x_0 + f(x_0, y_0)\Delta t$$
$$= 2 + 2y_0 \cdot 1$$
$$= 2 + 2\cdot 3 \cdot 1 = 8$$

$$y_1 = y_0 + g(x_0, y_0)\cdot \Delta t$$
$$= 3 + (-x_0)\cdot \Delta t$$
$$= 3 - 2 = 1$$



$$x_2 = x_1 + f(x_1, y_1)\Delta t$$
$$= 8 + 2y_1$$
$$= 8 + 2 = 10$$

$$y_2 = y_1 + g(x_1, y_1)\cdot \Delta t$$
$$= 8 + (-x_1)\cdot 1$$
$$= 1 - 1 = 0$$

$(x_2, y_2) = (10, 0)$

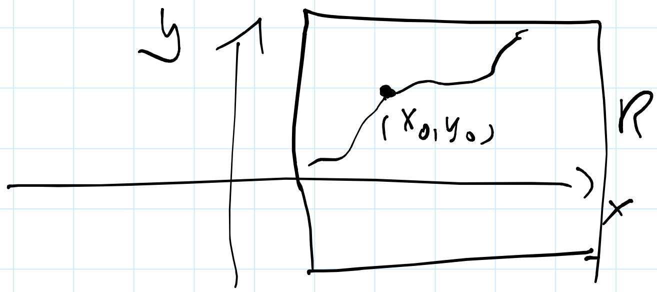
Ex & Uniqueness theorem

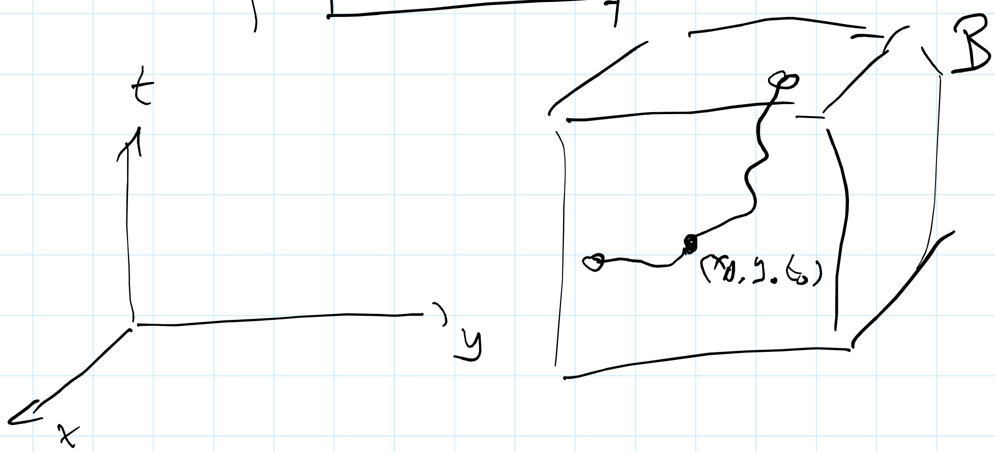
— for autonomous systems

— for autonomous systems

$$x' = f(x,y) \qquad x(0) = x_0$$
$$y' = g(x,y) \qquad y(0) = y_0$$

If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$ are continuous in a rectangle R containing (x_0, y_0), there will be a unique solution to the IVP as long as it stays in the rectangle





Phase portraits cannot intersect for autonomous systems!

→ follows from time-shift invariance of solutions to an autonomous system:

If $(x(t), y(t))$ is a solution

If $(x(t), y(t))$ is a solution
then $(x(t-t_0), y(t-t_0))$
will also be a solution
for all t_0!

Glider Example : Worksheet 2